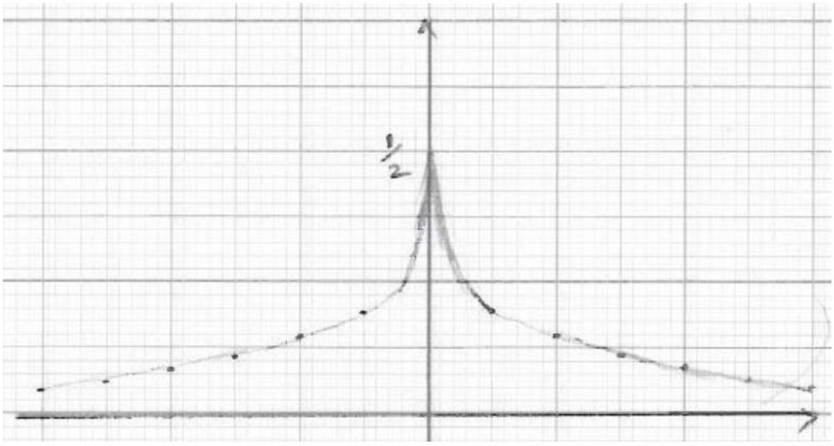
STEP III, 2008 Q12 MS

12.



$$M_x(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f(x)dx = \int_{-\infty}^{0} \frac{1}{2} e^{x(1+\theta)}dx + \int_{0}^{\infty} \frac{1}{2} e^{-x(1-\theta)}dx$$

$$= \frac{1}{2(1+\theta)}\left[e^{x(1+\theta)}\right]_{-\infty}^{0} - \frac{1}{2(1-\theta)}\left[e^{-x(1-\theta)}\right]_{0}^{\infty} \quad = \frac{1}{2(1+\theta)} + \frac{1}{2(1-\theta)} \quad \text{(requiring } |\theta| < 1\text{)}$$

$$= \left(1-\theta^2\right)^{-1}$$

$$Var(X) = M_x{}''(0) - \left(M_x{}'(0)\right)^2$$

$$M_x{}'(\theta) = 2\theta\left(1-\theta^2\right)^{-2}, \; M_x{}''(\theta) = 2\left(1-\theta^2\right)^{-2} + 8\theta\left(1-\theta^2\right)^{-3}$$

and so $M_x{}'(0) = 0$, $M_x{}''(0) = 2$, $Var(X) = 2$

Or alternatively, $M_x(\theta) = E\left(e^{\theta X}\right) = E\left(1 + \theta X + \frac{1}{2}\theta^2 X^2 + ...\right)$

$$= \left(1-\theta^2\right)^{-1} = 1 + \theta^2 + \theta^4 + ...$$

and so $E(X) = 0$, $E\left(\frac{1}{2}X^2\right) = 1$, $Var(X) = E\left(X^2\right) - \left(E(X)\right)^2 = 2$

If $T = Y/\sqrt{2n}$, then $M_T(\theta) = E\left(e^{\theta T}\right) = E\left(e^{\theta\sum\left(X_i/\sqrt{2n}\right)}\right) = \prod_{i=1}^{n} E\left(e^{\frac{\theta}{\sqrt{2n}}X_i}\right) = \left(1 - \frac{\theta^2}{2n}\right)^{-n}$

$$\log\left(M_T(\theta)\right) = -n\log\left(1-\frac{\theta^2}{2n}\right) = -n\left[-\frac{\theta^2}{2n}-\frac{\theta^4}{8n^2}-\frac{\theta^6}{24n^3}-.....\right] = \frac{\theta^2}{2}+\frac{\theta^4}{8n}+\frac{\theta^6}{24n^2}+.....$$

Thus as $n \to \infty$, $\log\left(M_T(\theta)\right) \to \frac{\theta^2}{2}$, and so $M_T(\theta) \to \exp\left(\frac{\theta^2}{2}\right)$

$P(|Y| \geq 25) = 0\cdot05$ and $P\left(\left|Y/\sqrt{2n}\right| \geq 1\cdot96\right) = 0\cdot05$ and so

$$25 = 1\cdot96\sqrt{2n}$$

$$2n = \frac{25^2}{1\cdot96^2} \approx \frac{625}{4}$$

$$n \approx \frac{625}{8} \approx 78$$