STEP III, 2008 Q10 MS

10. Considering the rth short string $T_r = mg + T_{r-1}$

Also we have $T_r = \frac{\lambda x_r}{l}$, and $T_1 = mg$

Thus $T_r = rmg$ and so the total length is given by $\sum_1^n (l + x_r) = \sum_1^n \left(l + \frac{rmgl}{\lambda} \right)$

$= nl + \frac{mgl}{\lambda} \frac{n(n+1)}{2}$

The elastic energy stored is $\sum_1^n \frac{\lambda x_r^2}{2l} = \sum_1^n \lambda \left(\frac{lmg}{\lambda} \right)^2 \frac{r^2}{2l} = \frac{m^2 g^2 l}{12\lambda} n(n+1)(2n+1)$

For the uniform heavy rope, we let $M = nm$, $L_0 = nl$, and consider the limit as $n \to \infty$

$L = \lim \left(L_0 + \frac{M}{n} \frac{g}{\lambda} \frac{L_0}{n} \frac{n(n+1)}{2} \right) = L_0 \left(1 + \frac{Mg}{2\lambda} \right)$

and the elastic energy stored is

$\lim \left(\frac{m^2 g^2 l}{12\lambda} n(n+1)(2n+1) \right) = \lim \left(\frac{M^2 g^2 L_0}{12\lambda} \frac{n(n+1)(2n+1)}{n^3} \right) = \frac{M^2 g^2 L_0}{6\lambda}$

and eliminating M using the result just found for L we obtain $\frac{2\lambda (L - L_0)^2}{3L_0}$