STEP III, 2007 Q9 MS

9. Conservation of energy leads to the equation

$2\left[\frac{1}{2}m\left(a\dot{\theta}\right)^2\right]+mk^2a^2(\theta-\alpha)^2=mk^2a^2(\beta-\alpha)^2$ which, when simplified, and working in the variable $(\theta-\alpha)$ rather than θ can be rearranged as

$(\dot{\theta}-\alpha)=k\sqrt{\left((\beta-\alpha)^2-(\theta-\alpha)^2\right)}$.

Separating the variables and performing the standard integral yields

$\theta-\alpha=(\beta-\alpha)\sin(kt+\phi)$ (it does not matter that $(\beta-\alpha)<0$).

The initial position from which the system is released gives $\phi=\frac{\pi}{2}$ and so

$\theta=\alpha+(\beta-\alpha)\cos kt$.

The three possibilities that can arise are that $\dot{\theta}=0, \theta<\frac{\pi}{2}$, that $\dot{\theta}=0, \theta=\frac{\pi}{2}$, or that

$\dot{\theta}>0, \theta=\frac{\pi}{2}$.

The first of these is SHM and has period $\frac{2\pi}{k}$, which occurs if $\alpha-(\beta-\alpha)<\frac{\pi}{2}$

i.e. if $\beta>2\alpha-\frac{\pi}{2}$.

For the second case, oscillations do not occur. Then,

$\dot{\theta}=0 \Rightarrow \sin kt=0 \Rightarrow \cos kt=-1$ (not $\cos kt=1$ as this is the initial position) and so

$\frac{\pi}{2}=\alpha-(\beta-\alpha)$ i.e. $\beta=2\alpha-\frac{\pi}{2}$.

The third case is partially SHM until $\theta=\frac{\pi}{2}$ and then the motion is reflected.

So a quarter of the period is given by $\frac{\pi}{2}=\alpha+(\beta-\alpha)\cos kt$ and hence the period is

$\frac{4}{k}\cos^{-1}\left(\frac{\frac{\pi}{2}-\alpha}{\beta-\alpha}\right)$ which occurs if $\beta<2\alpha-\frac{\pi}{2}$.