

## STEP III, 2007 Q9 MS

9. Conservation of energy leads to the equation

$2\left[\frac{1}{2}m(a\dot{\theta})^2\right] + mk^2a^2(\theta - \alpha)^2 = mk^2a^2(\beta - \alpha)^2$  which, when simplified, and

working in the variable  $(\theta - \alpha)$  rather than  $\theta$  can be rearranged as

$$(\theta - \alpha) = k\sqrt{((\beta - \alpha)^2 - (\theta - \alpha)^2)}.$$

Separating the variables and performing the standard integral yields

$$\theta - \alpha = (\beta - \alpha)\sin(kt + \phi) \text{ (it does not matter that } (\beta - \alpha) < 0\text{)}.$$

The initial position from which the system is released gives  $\phi = \frac{\pi}{2}$  and so

$$\theta = \alpha + (\beta - \alpha)\cos kt.$$

The three possibilities that can arise are that  $\dot{\theta} = 0, \theta < \frac{\pi}{2}$ , that  $\dot{\theta} = 0, \theta = \frac{\pi}{2}$ , or that

$$\dot{\theta} > 0, \theta = \frac{\pi}{2}.$$

The first of these is SHM and has period  $\frac{2\pi}{k}$ , which occurs if  $\alpha - (\beta - \alpha) < \frac{\pi}{2}$

$$\text{i.e. if } \beta > 2\alpha - \frac{\pi}{2}.$$

For the second case, oscillations do not occur. Then,

$\dot{\theta} = 0 \Rightarrow \sin kt = 0 \Rightarrow \cos kt = -1$  (not  $\cos kt = 1$  as this is the initial position) and so

$$\frac{\pi}{2} = \alpha - (\beta - \alpha) \text{ i.e. } \beta = 2\alpha - \frac{\pi}{2}.$$

The third case is partially SHM until  $\theta = \frac{\pi}{2}$  and then the motion is reflected.

So a quarter of the period is given by  $\frac{\pi}{2} = \alpha + (\beta - \alpha)\cos kt$  and hence the period is

$$\frac{4}{k} \cos^{-1}\left(\frac{\frac{\pi}{2} - \alpha}{\beta - \alpha}\right) \text{ which occurs if } \beta < 2\alpha - \frac{\pi}{2}.$$



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