STEP III, 2007 Q8 MS

8. (i) Substituting each u into the differential equation yields simultaneous equations $a(x)+xb(x)=0$ and $e^{-x}(1-a(x)+b(x))=0$ which solve to give

$a(x)=\dfrac{x}{1+x}$ and $b(x)=\dfrac{-1}{1+x}$

The general solution is $u = Ax + Be^{-x}$.

$y=\dfrac{1}{3u}\dfrac{du}{dx} \Rightarrow \dfrac{dy}{dx}=\dfrac{-1}{3u^2}\left(\dfrac{du}{dx}\right)^2+\dfrac{1}{3u}\dfrac{d^2u}{dx^2}$ which when substituted into equation (*), multiplied by 3u, and collected on one side gives the required result.

$$u = Ax + Be^{-x} \Rightarrow \frac{du}{dx} = A - Be^{-x} \Rightarrow y = \frac{A - Be^{-x}}{3(Ax + Be^{-x})},$$

and substitution of $x=0, y=0$ gives $A=B$ and hence $y=\dfrac{1-e^{-x}}{3(x+e^{-x})}$.

(ii) Substituting $y=\dfrac{1}{u}\dfrac{du}{dx}$ into the given equation yields

$\dfrac{d^2u}{dx^2}+\dfrac{x}{1-x}\dfrac{du}{dx}-\dfrac{1}{1-x}u=0$ which is the equation in the first part with x replaced by $-x$

So the general solution is $u = Cx + De^{x}$

Substitution of $x=0, y=2$ again gives $A=B$, and hence $y=\dfrac{1+e^x}{x+e^x}$