STEP III, 2007, Q8

8 **(i)** Find functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$ such that $u = x$ and $u = \mathrm{e}^{-x}$ both satisfy the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \mathrm{a}(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{b}(x)u = 0\,.$$

For these functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$, write down the general solution of the equation.

Show that the substitution $y = \dfrac{1}{3u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$ transforms the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \qquad (*)$$

into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1+x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{1+x}u = 0$$

and hence show that the solution of equation $(*)$ that satisfies $y = 0$ at $x = 0$ is given by $y = \dfrac{1-\mathrm{e}^{-x}}{3(x+\mathrm{e}^{-x})}$.

(ii) Find the solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x}$$

that satisfies $y = 2$ at $x = 0$.