

STEP III, 2007 Q7 MS

$$7. \quad (i) \quad u = v^{-1} \Rightarrow \frac{du}{dv} = -v^{-2} \text{ so } t(x) = \int_{\infty}^{\frac{1}{x}} \frac{1}{1+v^{-2}} \times -v^{-2} dv = \int_{\frac{1}{x}}^{\infty} \frac{1}{v^2+1} dv$$

$$\text{so } t\left(\frac{1}{x}\right) + t(x) = \int_0^{\frac{1}{x}} \frac{1}{1+u^2} du + \int_{\frac{1}{x}}^{\infty} \frac{1}{v^2+1} dv = \int_0^{\infty} \frac{1}{1+u^2} du = \frac{1}{2} p$$

Letting $x = 1$ gives the desired result.

$$(ii) \quad y = \frac{u}{\sqrt{1+u^2}} \Rightarrow u = \frac{y}{\sqrt{1-y^2}}$$

$$\text{so } \frac{du}{dy} = \frac{(1-y^2)^{\frac{1}{2}} - y \times -y(1-y^2)^{-\frac{1}{2}}}{1-y^2} = \frac{(1-y^2) + y^2}{(1-y^2)^{\frac{3}{2}}} \text{ and hence the result.}$$

Using the given substitution for u ,

$$t(x) = \int_0^{\frac{x}{\sqrt{1+x^2}}} \frac{1}{1+\frac{y^2}{1-y^2}} \times \frac{1}{(1-y^2)^{\frac{3}{2}}} dy = \int_0^{\frac{x}{\sqrt{1+x^2}}} \frac{1}{(1-y^2)^{\frac{1}{2}}} dy = s\left(\frac{x}{\sqrt{1+x^2}}\right)$$

Again letting $x = 1$, and using the result from part (i) gives the desired result.

$$(iii) \quad z = \frac{u + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}u} \Rightarrow u = \frac{z - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}z} \Rightarrow \frac{du}{dz} = \frac{\frac{4}{3}}{\left(1 + \frac{1}{\sqrt{3}}z\right)^2}$$

Using this substitution,

$$t(x) = \int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}} \frac{1}{1+\left(\frac{z-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}z}\right)^2} \times \frac{\frac{4}{3}}{\left(1+\frac{1}{\sqrt{3}}z\right)^2} dz = \int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}} \frac{\frac{4}{3}}{\left(1+\frac{1}{\sqrt{3}}z\right)^2 + \left(z-\frac{1}{\sqrt{3}}\right)^2} dz = \int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}} \frac{1}{1+z^2} dz$$

Letting $x = \frac{1}{\sqrt{3}}$ gives the required result.



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By definition $t\left(\frac{1}{\sqrt{3}}\right) = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+u^2} du$, by the previous result just obtained

$t\left(\frac{1}{\sqrt{3}}\right) = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2} dz$, and from part (i) $t\left(\frac{1}{\sqrt{3}}\right) = \int_{\sqrt{3}}^{\infty} \frac{1}{1+v^2} dv$ and so adding these three

results gives $3t\left(\frac{1}{\sqrt{3}}\right) = \int_0^{\infty} \frac{1}{1+u^2} du = \frac{1}{2} \pi$



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