STEP III, 2007 Q7 MS

7. (i) $u = v^{-1} \Rightarrow \dfrac{du}{dv} = -v^{-2}$ so $t(x) = \displaystyle\int_{\infty}^{\frac{1}{x}} \frac{1}{1+v^{-2}} \times -v^{-2}\,dv = \int_{\frac{1}{x}}^{\infty} \frac{1}{v^2+1}\,dv$

so $t\left(\dfrac{1}{x}\right) + t(x) = \displaystyle\int_0^{\frac{1}{x}} \frac{1}{1+u^2}\,du + \int_{\frac{1}{x}}^{\infty} \frac{1}{v^2+1}\,dv = \int_0^{\infty} \frac{1}{1+u^2}\,du = \frac{1}{2}p$

Letting $x = 1$ gives the desired result.

(ii) $y = \dfrac{u}{\sqrt{1+u^2}} \Rightarrow u = \dfrac{y}{\sqrt{1-y^2}}$

so $\dfrac{du}{dy} = \dfrac{(1-y^2)^{\frac{1}{2}} - y \times -y(1-y^2)^{\frac{-1}{2}}}{1-y^2} = \dfrac{(1-y^2)+y^2}{(1-y^2)^{\frac{3}{2}}}$ and hence the result.

Using the given substitution for u,

$t(x) = \displaystyle\int_0^{\frac{x}{\sqrt{1+x^2}}} \frac{1}{1+\frac{y^2}{1-y^2}} \times \frac{1}{(1-y^2)^{\frac{3}{2}}}\,dy = \int_0^{\frac{x}{\sqrt{1+x^2}}} \frac{1}{(1-y^2)^{\frac{1}{2}}}\,dy = s\left(\frac{x}{\sqrt{1+x^2}}\right)$

Again letting $x = 1$, and using the result from part (i) gives the desired result.

(iii) $z = \dfrac{u + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}u} \Rightarrow u = \dfrac{z - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}z} \Rightarrow \dfrac{du}{dz} = \dfrac{\frac{4}{3}}{\left(1 + \frac{1}{\sqrt{3}}z\right)^2}$

Using this substitution,

$t(x) = \displaystyle\int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}} \frac{1}{1+\left(\frac{z-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}z}\right)^2} \times \frac{\frac{4}{3}}{\left(1+\frac{1}{\sqrt{3}}z\right)^2}\,dz = \int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}} \frac{\frac{4}{3}}{\left(1+\frac{1}{\sqrt{3}}z\right)^2 + \left(z-\frac{1}{\sqrt{3}}\right)^2}\,dz = \int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}} \frac{1}{1+z^2}\,dz$

Letting $x = \dfrac{1}{\sqrt{3}}$ gives the required result.

By definition $t\left(\frac{1}{\sqrt{3}}\right) = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+u^2}\,du$, by the previous result just obtained

$t\left(\frac{1}{\sqrt{3}}\right) = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2}\,dz$, and from part (i) $t\left(\frac{1}{\sqrt{3}}\right) = \int_{\sqrt{3}}^{\infty} \frac{1}{1+v^2}\,dv$ and so adding these three

results gives $3t\left(\frac{1}{\sqrt{3}}\right) = \int_0^{\infty} \frac{1}{1+u^2}\,du = \frac{1}{2}p$