

STEP III, 2007, Q7

- 7 The functions $s(x)$ ($0 \leq x < 1$) and $t(x)$ ($x \geq 0$), and the real number p , are defined by

$$s(x) = \int_0^x \frac{1}{\sqrt{1-u^2}} du, \quad t(x) = \int_0^x \frac{1}{1+u^2} du, \quad p = 2 \int_0^\infty \frac{1}{1+u^2} du.$$

For this question, do not evaluate any of the above integrals explicitly in terms of inverse trigonometric functions or the number π .

- (i) Use the substitution $u = v^{-1}$ to show that $t(x) = \int_{1/x}^\infty \frac{1}{1+v^2} dv$. Hence evaluate $t(1/x) + t(x)$ in terms of p and deduce that $2t(1) = \frac{1}{2}p$.

- (ii) Let $y = \frac{u}{\sqrt{1+u^2}}$. Express u in terms of y , and show that $\frac{du}{dy} = \frac{1}{\sqrt{(1-y^2)^3}}$.

By making a substitution in the integral for $t(x)$, show that

$$t(x) = s\left(\frac{x}{\sqrt{1+x^2}}\right).$$

Deduce that $s\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}p$.

- (iii) Let $z = \frac{u + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}u}$. Show that $t\left(\frac{1}{\sqrt{3}}\right) = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2} dz$, and hence that $3t\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2}p$.



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