STEP III, 2007, Q7

7 The functions $\mathrm{s}(x)$ $(0 \leqslant x < 1)$ and $\mathrm{t}(x)$ $(x \geqslant 0)$, and the real number p, are defined by

$$\mathrm{s}(x) = \int_0^x \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u\,, \qquad \mathrm{t}(x) = \int_0^x \frac{1}{1+u^2}\,\mathrm{d}u\,, \qquad p = 2\int_0^\infty \frac{1}{1+u^2}\,\mathrm{d}u\,.$$

For this question, do not evaluate any of the above integrals explicitly in terms of inverse trigonometric functions or the number π.

(i) Use the substitution $u = v^{-1}$ to show that $\mathrm{t}(x) = \displaystyle\int_{1/x}^\infty \frac{1}{1+v^2}\,\mathrm{d}v$. Hence evaluate $\mathrm{t}(1/x) + \mathrm{t}(x)$ in terms of p and deduce that $2\mathrm{t}(1) = \frac{1}{2}p$.

(ii) Let $y = \dfrac{u}{\sqrt{1+u^2}}$. Express u in terms of y, and show that $\dfrac{\mathrm{d}u}{\mathrm{d}y} = \dfrac{1}{\sqrt{(1-y^2)^3}}$.

By making a substitution in the integral for $\mathrm{t}(x)$, show that

$$\mathrm{t}(x) = \mathrm{s}\left(\frac{x}{\sqrt{1+x^2}}\right).$$

Deduce that $\mathrm{s}(\frac{1}{\sqrt{2}}) = \frac{1}{4}p$.

(iii) Let $z = \dfrac{u + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}u}$. Show that $\mathrm{t}(\frac{1}{\sqrt{3}}) = \displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2}\,\mathrm{d}z$, and hence that $3\mathrm{t}(\frac{1}{\sqrt{3}}) = \frac{1}{2}p$.