STEP III, 2007 Q6 MS

6. $pp^* = qq^* = a^2$

and so $a^2(p-q) = qq^*p - pp^*q = -pq(p^* - q^*)$ and hence the required result.

If PQ and RS are perpendicular then $p - q = ki(r - s)$ for some real k, and thus

$p^* - q^* = -ki(r^* - s^*)$, and so $pq = -a^2 \dfrac{p-q}{p^*-q^*} = a^2 \dfrac{r-s}{r^*-s^*} = -rs$

For $n = 3$, $B_1B_2 \perp A_1A_2$ etc. $\Rightarrow a_1a_2 + b_1b_2 = 0$ etc.

Thus $b_1^{\ 2} = \dfrac{b_1b_2 \times b_1b_3}{b_2b_3} = \dfrac{-a_1a_2 \times -a_1a_3}{-a_2a_3} = -a_1^{\ 2}$ and so $b_1 = \pm ia_1$

i.e. two choices of B_1.

For $n = 4$, $B_1B_2 \perp A_1A_2$ etc. $\Rightarrow a_1a_2 + b_1b_2 = 0$ etc. but this only yields 3 independent equations as e.g. $a_3a_4 + b_3b_4 = 0$ can be obtained from the other three equations by $a_3a_4 = \dfrac{a_2a_3 \times a_4a_1}{a_2a_1}$ etc. Hence there are arbitrarily many possible choices for B_1.

For $n > 4$, the corresponding results are as for $n = 3$ or $n = 4$ depending on whether n is odd or even.