

### STEP III, 2007 Q5 MS

$$5. \quad \frac{dr}{dx} = x(x^2 - 1)^{-\frac{1}{2}} = \cosh \theta$$

$$y = \ln r^2 = 2 \ln r$$

$$\text{So } \frac{dy}{dx} = \frac{2}{r} \frac{dr}{dx} = \frac{2 \cosh \theta}{r}$$

$$\frac{dx}{d\theta} = -\operatorname{cosech}^2 \theta \text{ and } r = \operatorname{cosech} \theta,$$

So differentiating the previous result and substituting,

$$\frac{d^2 y}{dx^2} = \frac{2r \sinh \theta \frac{d\theta}{dx} - 2 \cosh \theta \frac{dr}{dx}}{r^2} = \frac{2(\operatorname{cosech} \theta \sinh \theta \times -\sinh^2 \theta - \cosh \theta \cosh \theta)}{r^2} = -\frac{2 \cosh 2\theta}{r^2}$$

Similarly,

$$\frac{d^3 y}{dx^3} = -\frac{2r^2 2 \sinh 2\theta \frac{d\theta}{dx} - 2 \cosh 2\theta \times 2r \frac{dr}{dx}}{r^4} = \frac{4}{r^4} (\sinh 2\theta + \cosh 2\theta \coth \theta) = \frac{4}{r^3} \cosh 3\theta$$

In order to hypothesise a result for  $\frac{d^n y}{dx^n}$ , the important thing is to appreciate that the 4 has come from 2 times exponent of  $r$  and multiple of  $\theta$ .

So  $\frac{d^n y}{dx^n} = 2 \times (-1)^{n-1} \frac{(n-1)!}{r^n} \cosh n\theta$  which may be proved by induction, the

inductive differentiation step following the same pattern of working as used for  $\frac{d^2 y}{dx^2}$

and  $\frac{d^3 y}{dx^3}$ .



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