STEP III, 2007 Q5 MS

5. $$\frac{dr}{dx} = x\left(x^2 - 1\right)^{\frac{-1}{2}} = \cosh\theta$$

$$y = \ln r^2 = 2\ln r$$

So $\dfrac{dy}{dx} = \dfrac{2}{r}\dfrac{dr}{dx} = \dfrac{2\cosh\theta}{r}$

$\dfrac{dx}{d\theta} = -\operatorname{cosech}^2\theta$ and $r = \operatorname{cosech}\theta$,

So differentiating the previous result and substituting,

$$\frac{d^2y}{dx^2} = \frac{2r\sinh\theta\dfrac{d\theta}{dx} - 2\cosh\theta\dfrac{dr}{dx}}{r^2} = \frac{2\left(\operatorname{cosech}\theta\sinh\theta\times -\sinh^2\theta - \cosh\theta\cosh\theta\right)}{r^2} = -\frac{2\cosh 2\theta}{r^2}$$

Similarly,

$$\frac{d^3y}{dx^3} = -\frac{2r^2 2\sinh 2\theta\dfrac{d\theta}{dx} - 2\cosh 2\theta\times 2r\dfrac{dr}{dx}}{r^4} = \frac{4}{r^4}\left(\sinh 2\theta + \cosh 2\theta\coth\theta\right) = \frac{4}{r^3}\cosh 3\theta$$

In order to hypothesise a result for $\dfrac{d^ny}{dx^n}$, the important thing is to appreciate that the 4 has come from 2 times exponent of r and multiple of θ.

So $\dfrac{d^ny}{dx^n} = 2\times(-1)^{n-1}\dfrac{(n-1)!}{r^n}\cosh n\theta$ which may be proved by induction, the inductive differentiation step following the same pattern of working as used for $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$.