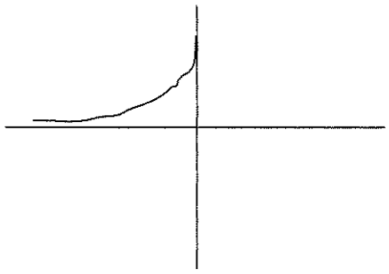
STEP III, 2007 Q4 MS

4.

$y = a\sin t \Rightarrow \dot{y} = a\cos t$

$$x = a\left(\cos t + \ln\tan\frac{t}{2}\right) \Rightarrow \dot{x} = a\left(-\sin t + \frac{\frac{1}{2}\sec^2\frac{t}{2}}{\tan\frac{t}{2}}\right) = a(-\sin t + \operatorname{cosec} t) = a\cos t\cot t$$

giving $\frac{dy}{dx} = \tan t$.



(y intercept a, y axis tangential to curve, x axis asymptote)

Tangent is $y - a\sin t = \tan t\left(x - a\left(\cos t + \ln\tan\frac{t}{2}\right)\right)$ giving Q as $\left(a\ln\tan\frac{t}{2}, 0\right)$ and

thus $PQ = \sqrt{\left((a\cos t)^2 + (a\sin t)^2\right)} = a$

$\dot{y} = a\cos t \Rightarrow \ddot{y} = -a\sin t$

$\dot{x} = a(-\sin t + \operatorname{cosec} t) \Rightarrow \ddot{x} = a(-\cos t - \operatorname{cosec} t\cot t)$

$\dot{x}^2 + \dot{y}^2 = (a\cos t\cot t)^2 + (a\cos t)^2 = a^2\cot^2 t$

$\dot{x}\ddot{y} - \dot{y}\ddot{x} = a\cos t\cot t \times -a\sin t - a\cos t \times a(-\cos t - \operatorname{cosec} t\cot t)$

$= a^2\left(-\cos^2 t + \cos^2 t + \cot^2 t\right) = a^2\cot^2 t$

giving $\rho = a\cot t$.

From the results for $\frac{dy}{dx}$ and ρ, C is

$$\left(a\left(\cos t + \ln\tan\frac{t}{2}\right) - \rho\sin t, a\sin t + \rho\cos t\right) = \left(a\ln\tan\frac{t}{2}, a\operatorname{cosec} t\right)$$

Which has the same x coordinate as Q.