Attempt this question independently, showing full and clear working.

STEP III 2007 Question 4

A curve is given parametrically by

$$x = a\big(\cos t + \ln\tan\tfrac{1}{2}t\big), \qquad y = a\sin t,$$

where $0 < t < \frac{1}{2}\pi$ and a is a positive constant. Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \tan t$ and sketch the curve.

Let P be the point with parameter t and let Q be the point where the tangent to the curve at P meets the x-axis. Show that $PQ = a$.

The *radius of curvature*, ρ, at P is defined by

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|},$$

where the dots denote differentiation with respect to t. Show that $\rho = a\cot t$.

The point C lies on the normal to the curve at P, a distance ρ from P and above the curve. Show that CQ is parallel to the y-axis.