STEP III, 2007, Q4

4 A curve is given parametrically by

$$x = a\left(\cos t + \ln \tan \tfrac{1}{2}t\right),$$
$$y = a \sin t,$$

where $0 < t < \frac{1}{2}\pi$ and a is a positive constant. Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \tan t$ and sketch the curve.

Let P be the point with parameter t and let Q be the point where the tangent to the curve at P meets the x-axis. Show that $PQ = a$.

The *radius of curvature*, ρ, at P is defined by

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|},$$

where the dots denote differentiation with respect to t. Show that $\rho = a \cot t$.

The point C lies on the normal to the curve at P, a distance ρ from P and above the curve. Show that CQ is parallel to the y-axis.