STEP III, 2007 Q3 MS

3. (i) $F_3 = 2, F_4 = 3, F_6 = 5, F_6 = 8, F_7 = 13, F_8 = 21$

(ii) The result requires no term beyond F_{2k+2} should appear on the RHS so the first strategy is to replace F_{2k+3} and hence

$$F_{2k+3}F_{2k+1} - F_{2k+2}{}^2 = \left(F_{2k+2} + F_{2k+1}\right)F_{2k+1} - F_{2k+2}{}^2 = \left(F_{2k+1} - F_{2k+2}\right)F_{2k+2} + F_{2k+1}{}^2 = -F_{2k}F_{2k+2} + F_{2k+1}{}^2$$

as required.

(iii) The initial case is trivial to demonstrate, and so the induction runs from assuming that $F_{2k+1}F_{2k-1} - F_{2k}{}^2 = 1$, and attempting to prove that

$F_{2(k+1)+1}F_{2(k+1)-1} - F_{2(k+1)}{}^2 = 1$.

$F_{2(k+1)+1}F_{2(k+1)-1} - F_{2(k+1)}{}^2 = F_{2k+3}F_{2k+1} - F_{2k+2}{}^2 = -F_{2k}F_{2k+2} + F_{2k+1}{}^2$ from (ii)

$= -\left(-F_{2k-1}F_{2k+1} + F_{2k}{}^2\right)$ by a similar argument to (ii) $= -(-1)$ by inductive hypothesis.

The deduction follows from adding $F_{2n}{}^2$ to both sides of the result just proved.

(iv) This result cannot be deduced directly from (iii) as the nature of the expression differs in the type of subscript. Thus consider

$$F_{2n-1}{}^2 + 1 = \left(F_{2n+1} - F_{2n}\right)^2 + 1 = F_{2n+1}{}^2 - 2F_{2n+1}F_{2n} + F_{2n}{}^2 + 1 = F_{2n+1}{}^2 - 2F_{2n+1}F_{2n} + F_{2n-1}F_{2n+1}$$

from (iii) and hence the desired result is obtained.