# STEP III, 2007 Q2 MS

2. (i)

$$1.3.5.7.....(2n-1) = \frac{1.2.3.4.....2n}{2.4.6.8.....2n} == \frac{(2n)!}{2.1.2.2.2.3.2.4.....2.n} = \frac{(2n)!}{2^n.1.2.3.4.....n} = \frac{(2n)!}{2^n n!}$$

Using the binomial theorem, which is valid given the condition $|x| < \frac{1}{4}$,

$$(1-4x)^{\frac{-1}{2}} = 1 + \frac{-1}{2}(-4x) + \frac{\frac{-1}{2}\frac{-3}{2}(-4x)^2}{2!} + \ldots$$

$$= 1 + 1.(2x) + \frac{1.3}{2!}(2x)^2 + \ldots + \frac{1.3.5.7\ldots(2n-1)}{n!}(2x)^n + \ldots$$

So the first result of the question yields $(1-4x)^{\frac{-1}{2}} = 1 + \sum_{n=1}^{\infty} \frac{\frac{(2n)!}{2^n n!}(2x)^n}{n!}$ leading to the required expression.

(ii) Differentiating $(1-4x)^{\frac{-1}{2}} = 1 + \sum_{n=1}^{\infty} \frac{(2n)!x^n}{(n!)^2}$ with respect to $x$, and multiplying the result by $x$ gives $\frac{2x}{(1-4x)^{\frac{3}{2}}} = \sum_{n=1}^{\infty} \frac{(2n)!x^n}{n!(n-1)!}$ and substituting $x = \frac{6}{25} < \frac{1}{4}$, gives the desired result.

(iii) Integrating $(1-4x)^{\frac{-1}{2}} = 1 + \sum_{n=1}^{\infty} \frac{(2n)!x^n}{(n!)^2}$ with respect to $x$, gives $\frac{-1}{2}(1-4x)^{\frac{1}{2}} = x + \sum_{n=1}^{\infty} \frac{(2n)!x^{n+1}}{(n+1)!n!} + c$, and substituting $x = 0 < \frac{1}{4}$, gives $c = \frac{-1}{2}$.

Now substituting $x = \frac{2}{9} = \frac{2}{3^2} < \frac{1}{4}$ and simplifying, gives the desired result.