STEP III, 2007 Q1 MS

1. The first result can be obtained by applying a compound angle formula to $\tan((\theta_1+\theta_2)+(\theta_3+\theta_4))$ and then repeating the application to each of $\tan(\theta_1+\theta_2)$ and $\tan(\theta_3+\theta_4)$ where they appear. On simplification, this gives

$$\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\frac{t_1+t_2+t_3+t_4-t_2t_3t_4-t_3t_4t_1-t_4t_1t_2-t_1t_2t_3}{1-t_1t_2-t_1t_3-t_1t_4-t_2t_3-t_2t_4-t_3t_4+t_1t_2t_3t_4}.$$

As t_1, etc are the roots of the equation $at^4+bt^3+ct^2+dt+e=0$, then $at^4+bt^3+ct^2+dt+e=a(t-t_1)(t-t_2)(t-t_3)(t-t_4)$, which yields, from expansion and comparison of coefficients, the four results

$$t_1+t_2+t_3+t_4=\frac{-b}{a},\quad t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4=\frac{c}{a},$$

$$t_2t_3t_4+t_3t_4t_1+t_4t_1t_2+t_1t_2t_3=\frac{-d}{a}, \text{ and } t_1t_2t_3t_4=\frac{e}{a}.$$

These substituted in the first result lead to $\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\dfrac{-b+d}{a-c+e}$.

Applying double and compound angle formulae to $p\cos 2\vartheta+\cos(\vartheta-\alpha)+p=0$ gives the equation $2p\cos^2\vartheta+\cos\vartheta\cos\alpha+\sin\vartheta\sin\alpha=0$, which can be rearranged as

$$\cos\alpha+\tan\vartheta\sin\alpha=\frac{-2p}{\sec\vartheta}.$$

Squaring this and replacing $\tan\vartheta$ by t, $(\cos\alpha+t\sin\alpha)^2=\dfrac{4p^2}{1+t^2}$.

Rearranging this obtains the quartic equation $t^4\sin^2\alpha+t^3\sin 2\alpha+t^2+t\sin 2\alpha+(\cos^2\alpha-4p^2)=0$, and so, from the second result $\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\dfrac{0}{-4p^2}=0$, and thus $\theta_1+\theta_2+\theta_3+\theta_4=n\pi$.