STEP III 2007 Question 14 — Mark Scheme

(i) If W is the area of the smallest circle with centre O that encloses the hole made by a single dart throw, then the p.d.f. of W is given by

$$f(w) = \begin{cases} \dfrac{1}{\pi}, & 0 \le w \le \pi \\ 0, & \text{otherwise.} \end{cases}$$

If X is the area of the smallest circle with centre O that encloses all the n holes made, then

$$P(x < X < x + \delta x) = n\left(\frac{x}{\pi}\right)^{n-1} \frac{\delta x}{\pi} \qquad \text{and so} \qquad E(X) = \int_0^{\pi} x \cdot n\left(\frac{x}{\pi}\right)^{n-1} \frac{1}{\pi}\, dx = \frac{n\pi}{n+1}.$$

On the other hand, if Y is the area of the smallest circle with centre O that encloses all the $(n-1)$ holes nearest to O, then

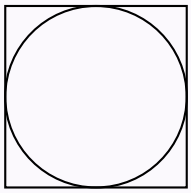
$$P(x < Y < x + \delta x) = n(n-1)\left(\frac{x}{\pi}\right)^{n-2}\left(1 - \frac{x}{\pi}\right)\frac{\delta x}{\pi}$$

and so

$$E(Y) = \int_0^{\pi} x \cdot n(n-1)\left[\left(\frac{x}{\pi}\right)^{n-2} - \left(\frac{x}{\pi}\right)^{n-1}\right]\frac{1}{\pi}\, dx = \frac{(n-1)\pi}{n+1}.$$

(ii) If Z is the area of the smallest square with centre Q that encloses all the n holes made, then, in similar manner to (i),

$$P(x < Z < x + \delta x) = n\left(\frac{x}{4}\right)^{n-1} \frac{\delta x}{4} \qquad \text{and so} \qquad E(Z) = \int_0^{4} x \cdot n\left(\frac{x}{4}\right)^{n-1} \frac{1}{4}\, dx = \frac{4n}{n+1}.$$

(iii) If we knew that the dart landed inside the circle of radius 1 centre Q when it hit the square dartboard, then the answer would be that we obtained for the circular board. But there is a non-zero probability that the dart could land in larger circles if it fell on the board outside the circle of radius 1, and hence the expected area of the smallest circle for the square dartboard is larger than that for the circular board.
Algebraically, if S is the expected area of such a circle if the dart falls outside the circle on the square board, and $E(X)$ is as in part (i),
the expected area $= \dfrac{\pi}{4}E(X) + \left(1 - \dfrac{\pi}{4}\right)S$, where $S > E(X)$, and so this is

$$\left(1 - \left(1 - \frac{\pi}{4}\right)\right)E(X) + \left(1 - \frac{\pi}{4}\right)S = E(X) + \left(1 - \frac{\pi}{4}\right)(S - E(X)) > E(X).$$