

## STEP III, 2007 Q14 MS

14. (i) If  $W$  is the area of the smallest circle with centre  $O$  that encloses the hole made by a single dart throw then the p.d.f. of  $W$  is given by

$$f(w) = \begin{cases} \frac{1}{\pi}, & 0 \leq w \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

If  $X$  is the area of the smallest circle with centre  $O$  that encloses all the  $n$  holes made then

$$P(x < X < x + \delta x) = n \left(\frac{x}{\pi}\right)^{n-1} \frac{\delta x}{\pi} \text{ and so } E(X) = \int_0^{\pi} x \times n \left(\frac{x}{\pi}\right)^{n-1} \frac{1}{\pi} dx = \frac{n\pi}{n+1}.$$

On the other hand, if  $Y$  is the area of the smallest circle with centre  $O$  that encloses all the  $(n-1)$  holes nearest to  $O$  then  $P(x < Y < x + \delta x) = n(n-1) \left(\frac{x}{\pi}\right)^{n-2} \left(1 - \frac{x}{\pi}\right) \frac{\delta x}{\pi}$  and

$$\text{so } E(Y) = \int_0^{\pi} x \times n(n-1) \left(\left(\frac{x}{\pi}\right)^{n-2} - \left(\frac{x}{\pi}\right)^{n-1}\right) \frac{1}{\pi} dx = \frac{(n-1)\pi}{n+1}$$

(ii) If  $Z$  is the area of the smallest square with centre  $Q$  that encloses all the  $n$  holes made then, in similar manner to (i)

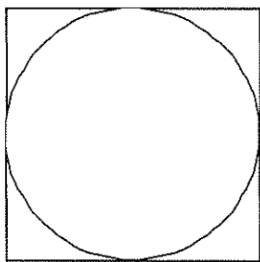
$$P(x < Z < x + \delta x) = n \left(\frac{x}{4}\right)^{n-1} \frac{\delta x}{4} \text{ and so } E(Z) = \int_0^4 x \times n \left(\frac{x}{4}\right)^{n-1} \frac{1}{4} dx = \frac{4n}{n+1}.$$

(iii) If we knew that the dart landed inside the circle of radius 1 centre  $Q$  when it hit the square dartboard, then the answer would be that we obtained for the circular board. But there is a non-zero probability that the dart could land in larger circles if it fell on the board outside the circle of radius 1 and hence the expected area of the smallest circle for the square dartboard is larger than that for the circular board.

Algebraically, if  $S$  is the expected area of such a circle if the dart falls outside the circle on the square board, and  $E(X)$  is as in part (i),

the expected area =  $\left(\frac{\pi}{4}\right)E(X) + \left(1 - \frac{\pi}{4}\right)S$ , where  $S > E(X)$ , and so this is

$$\left(1 - \left(1 - \frac{\pi}{4}\right)\right)E(X) + \left(1 - \frac{\pi}{4}\right)S = E(X) + \left(1 - \frac{\pi}{4}\right)(S - E(X)) > E(X)$$



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