STEP III, 2007 Q14 MS

14. (i) If W is the area of the smallest circle with centre O that encloses the hole made by a single dart throw then the p.d.f. of W is given by

$$f(w) = \begin{cases} \frac{1}{\pi}, 0 \le w \le \pi \\ 0, otherwise \end{cases}$$

If X is the area of the smallest circle with centre O that encloses all the n holes made then

$$P(x < X < x + \delta x) = n\left(\frac{x}{\pi}\right)^{n-1} \frac{\delta x}{\pi} \text{ and so } E(X) = \int_0^\pi x \times n\left(\frac{x}{\pi}\right)^{n-1} \frac{1}{\pi} dx = \frac{n\pi}{n+1}.$$

On the other hand, if Y is the area of the smallest circle with centre O that encloses all the $(n-1)$ holes nearest to O then $P(x < Y < x + \delta x) = n(n-1)\left(\frac{x}{\pi}\right)^{n-2}\left(1 - \frac{x}{\pi}\right)\frac{\delta x}{\pi}$ and

so $E(Y) = \int_0^\pi x \times n(n-1)\left(\left(\frac{x}{\pi}\right)^{n-2} - \left(\frac{x}{\pi}\right)^{n-1}\right)\frac{1}{\pi} dx = \frac{(n-1)\pi}{n+1}$

(ii) If Z is the area of the smallest square with centre Q that encloses all the n holes made then, in similar manner to (i)

$$P(x < Z < x + \delta x) = n\left(\frac{x}{4}\right)^{n-1} \frac{\delta x}{4} \text{ and so } E(Z) = \int_0^4 x \times n\left(\frac{x}{4}\right)^{n-1} \frac{1}{4} dx = \frac{4n}{n+1}.$$

(iii) If we knew that the dart landed inside the circle of radius 1 centre Q when it hit the square dartboard, then the answer would be that we obtained for the circular board. But there is a non-zero probability that the dart could land in larger circles if it fell on the board outside the circle of radius 1 and hence the expected area of the smallest circle for the square dartboard is larger than that for the circular board.

Algebraically, if S is the expected area of such a circle if the dart falls outside the circle on the square board, and $E(X)$ is as in part (i),

the expected area $= \left(\frac{\pi}{4}\right)E(X) + \left(1 - \frac{\pi}{4}\right)S$, where $S > E(X)$, and so this is

$$\left(1 - \left(1 - \frac{\pi}{4}\right)\right)E(X) + \left(1 - \frac{\pi}{4}\right)S = E(X) + \left(1 - \frac{\pi}{4}\right)(S - E(X)) > E(X)$$