STEP III, 2007 Q13 MS

13. (i) $p_2(2)$ is the probability of landing in the pool for the first time on the 2nd jump starting 1.5m away which is the probability that the first jump is 1m which is p.

(ii) $u_1 = 1$

$p_2(1) = q$ and $p_2(2) = p$ so $u_2 = q + 2p = 1 + p = 2 - q$

$p_3(1) = 0$, $p_3(2) = 1 - p^2 = q(1+p) = 2q - q^2$, and $p_3(3) = p^2 = 1 - 2q + q^2$ so

$u_3 = 2(2q - q^2) + 3(1 - 2q + q^2) = 3 - 2q + q^2$

(iii) Using the values $u_1 = 1$, $u_2 = 2 - q$, and $u_3 = 3 - 2q + q^2$, we obtain three equations:-

$A + B + C = 1$ (1)

$-Aq + B + 2C = 2 - q$ (2)

$Aq^2 + B + 3C = 3 - 2q + q^2$ (3)

It makes sense to consider $(3)-(2)$ and $(2)-(1)$ to eliminate B and then subtract the resulting equations to eliminate C, and hence we find that

$(3) - 2(2) + (1) \Rightarrow A(q^2 + 2q + 1) = q^2 \Rightarrow A = \left(\frac{q}{q+1}\right)^2$,

substituting in $(2)-(1) \Rightarrow \left(\frac{q}{q+1}\right)^2(-q-1) + C = 1 - q \Rightarrow C = \frac{1}{1+q}$, and so

$B = \frac{q}{(q+1)^2}$.

So $u_n = \left(\frac{q}{q+1}\right)^2(-q)^{n-1} + \frac{q}{(q+1)^2} + \frac{1}{1+q}n = \frac{(-q)^{n+1}}{(q+1)^2} + \frac{q}{(q+1)(p+2q)} + \frac{1}{p+2q}n$

For large n, the first term approaches zero, and the second term is negligible in comparison with the third for $\frac{q}{q+1} < 1 << n$

Hence $u_n \approx \frac{1}{p+2q}n$

The expected distance covered in one jump is $q + 2p$ and as jumps are of integer length, to get to the pool from a distance $\left(n - \frac{1}{2}\right)m$ needs a distance n metres to be jumped and so the expected number of jumps would be $\frac{1}{p+2q}n$.