STEP III, 2007, Q13

13 A frog jumps towards a large pond. Each jump takes the frog either 1 m or 2 m nearer to the pond. The probability of a 1 m jump is p and the probability of a 2 m jump is q, where $p + q = 1$, the occurence of long and short jumps being independent.

(i) Let $p_n(j)$ be the probability that the frog, starting at a point $(n - \frac{1}{2})$ m away from the edge of the pond, lands in the pond for the first time on its jth jump. Show that $p_2(2) = p$.

(ii) Let u_n be the expected number of jumps, starting at a point $(n - \frac{1}{2})$ m away from the edge of the pond, required to land in the pond for the first time. Write down the value of u_1. By finding first the relevant values of $p_n(m)$, calculate u_2 and show that $u_3 = 3 - 2q + q^2$.

(iii) Given that u_n can be expressed in the form $u_n = A(-q)^{n-1} + B + Cn$, where A, B and C are constants (independent of n), show that $C = (1+q)^{-1}$ and find A and B in terms of q. Hence show that, for large n, $u_n \approx \dfrac{n}{p + 2q}$ and explain carefully why this result is to be expected.