STEP III, 2007 Q12 MS

12.

$$E(N)=\sum_{i=1}^{2n-1}\frac{1}{2n-1}i=\frac{1}{2n-1}\frac{(2n-1)2n}{2}=n$$

$$E(N^2)=\sum_{i=1}^{2n-1}\frac{1}{2n-1}i^2=\frac{1}{2n-1}\frac{(2n-1)2n(4n-1)}{6}=\frac{n(4n-1)}{3}$$

$$E(Y)=E\left(\sum_{i=1}^{N}X_i\right)=\frac{1}{2n-1}E(X_1)+\frac{1}{2n-1}E(X_1+X_2)+\ldots..=\frac{1}{2n-1}(\mu+2\mu+3\mu+\ldots+(2n-1)\mu)$$

$$=\frac{1}{2n-1}\frac{\mu(2n-1)2n}{2}=n\mu$$

$$E(YN)=\frac{1}{2n-1}\times 1\times\mu+\frac{1}{2n-1}\times 2\times 2\mu+\ldots.+\frac{1}{2n-1}\times(2n-1)\times(2n-1)\mu=\frac{n(4n-1)}{3}\mu$$

and so $Cov(Y,N)=\dfrac{n(4n-1)}{3}\mu-n^2\mu=\dfrac{1}{3}n(n-1)\mu$

$$E(X_i^2)=Var(X_i)+(E(X_i))^2=\sigma^2+\mu^2$$

Also $(X_1+X_2+\ldots..+X_r)^2=\sum_{i=1}^{r}X_i^2+2\sum_{i\neq j}X_iX_j$, and so

$$E\left((X_1+X_2+\ldots..+X_r)^2\right)=r(\sigma^2+\mu^2)+2\frac{r(r-1)}{2}\mu^2$$

Thus

$$E(Y^2)=\frac{1}{2n-1}\sum_{r=1}^{2n-1}\left(r(\sigma^2+\mu^2)+2\frac{r(r-1)}{2}\mu^2\right)=n(\sigma^2+\mu^2)+\frac{n(4n-1)}{3}\mu^2-n\mu^2=n\sigma^2+\frac{n(4n-1)}{3}\mu^2$$

and so $Var(Y)=n\sigma^2+\dfrac{n(4n-1)}{3}\mu^2-n^2\mu^2=n\sigma^2+\dfrac{n(n-1)}{3}\mu^2$