STEP III, 2007 Q10 MS

10. Using uniform acceleration formulae with $(\ddot{x}, \ddot{y}) = (-g\sin\phi, -g\cos\phi)$, then

$(x, y) = \left(Vt\cos\theta - \frac{1}{2}gt^2\sin\phi, Vt\sin\theta - \frac{1}{2}gt^2\cos\phi\right)$.

To return on the same path $\dot{x} = 0$ when $y = 0$. So $t = \frac{V\cos\theta}{g\sin\phi} = \frac{2V\sin\theta}{g\cos\phi}$

i.e. $2\tan\phi\tan\theta = 1$

Also using $v^2 = u^2 + 2as$ in the x direction $0 = V^2\cos^2\theta - 2gR\sin\phi$

i.e. $R = \frac{V^2\cos^2\theta}{2g\sin\phi}$.

Thus

$\frac{2V^2}{gR} = 4\sin\phi\sec^2\theta = 4\sin\phi(1+\tan^2\theta) = 4\sin\phi\left(1+\frac{1}{4}\cot^2\phi\right) = 4\sin\phi\left(1+\frac{1}{4}(\operatorname{cosec}^2\phi - 1)\right)$

$= 3\sin\phi + \operatorname{cosec}\phi$

Consider $y = 3x + \frac{1}{x}, x > 0$. By differentiation, this is least for $x = \frac{1}{\sqrt{3}}$.

Thus the least value of $\frac{2V^2}{gR}$ is $2\sqrt{3}$, and the largest value of R is $\frac{V^2}{\sqrt{3}g}$.