STEP III, 2007, Q10

10 A particle is projected from a point on a plane that is inclined at an angle ϕ to the horizontal. The position of the particle at time t after it is projected is (x, y), where $(0, 0)$ is the point of projection, x measures distance up the line of greatest slope and y measures perpendicular distance from the plane. Initially, the velocity of the particle is given by $(\dot{x}, \dot{y}) = (V\cos\theta, V\sin\theta)$, where $V > 0$ and $\phi + \theta < \pi/2$. Write down expressions for x and y.

The particle bounces on the plane and returns along the same path to the point of projection. Show that

$$2\tan\phi\tan\theta = 1$$

and that

$$R = \frac{V^2\cos^2\theta}{2g\sin\phi},$$

where R is the range along the plane.

Show further that

$$\frac{2V^2}{gR} = 3\sin\phi + \operatorname{cosec}\phi$$

and deduce that the largest possible value of R is $V^2/(\sqrt{3}\,g)$.