STEP III, 2006, Q8

8 $\triangle$ is an operation that takes polynomials in x to polynomials in x; that is, given any polynomial $\mathrm{h}(x)$, there is a polynomial called $\triangle\mathrm{h}(x)$ which is obtained from $\mathrm{h}(x)$ using the rules that define $\triangle$. These rules are as follows:

(i) $\triangle x = 1$;

(ii) $\triangle\big(\mathrm{f}(x) + \mathrm{g}(x)\big) = \triangle\mathrm{f}(x) + \triangle\mathrm{g}(x)$ for any polynomials $\mathrm{f}(x)$ and $\mathrm{g}(x)$;

(iii) $\triangle\big(\lambda\mathrm{f}(x)\big) = \lambda\triangle\mathrm{f}(x)$ for any constant λ and any polynomial $\mathrm{f}(x)$;

(iv) $\triangle\big(\mathrm{f}(x)\mathrm{g}(x)\big) = \mathrm{f}(x)\triangle\mathrm{g}(x) + \mathrm{g}(x)\triangle\mathrm{f}(x)$ for any polynomials $\mathrm{f}(x)$ and $\mathrm{g}(x)$.

Using these rules show that, if $\mathrm{f}(x)$ is a polynomial of degree zero (that is, a constant), then $\triangle\mathrm{f}(x) = 0$. Calculate $\triangle x^2$ and $\triangle x^3$.

Prove that $\triangle\mathrm{h}(x) \equiv \dfrac{\mathrm{d}\mathrm{h}(x)}{\mathrm{d}x}$ for any polynomial $\mathrm{h}(x)$. You should make it clear whenever you use one of the above rules in your proof.