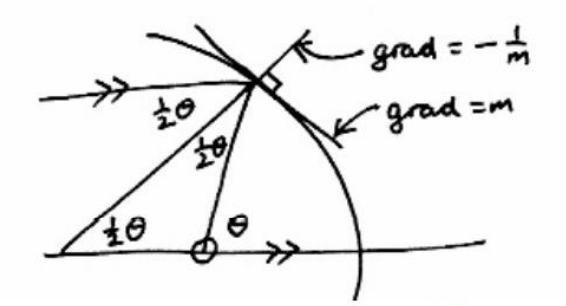
STEP III, 2006, Q6 MS

6

$x = r\cos\theta,\ y = r\sin\theta,\ r = r(\theta)$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

and result follows.



Gradient of the normal is $\tan\frac{\theta}{2} = t$, say. Then we have

$$t = -\frac{\frac{dr}{d\theta} - r\tan\theta}{\frac{dr}{d\theta}\tan\theta + r},\ \tan\theta = \frac{2t}{1-t^2}$$

This reduces to

$$\frac{dr}{d\theta} = rt$$

$$\Rightarrow \ln r = \int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} d\theta$$

$$= -2\ln\left[c\cos\frac{\theta}{2}\right]$$

$$\Rightarrow \frac{2}{c^2 r} = 1 + \cos\theta \quad (\text{using } 1+\cos\theta = 2\cos^2\frac{\theta}{2})$$

This corresponds to the standard equation of a parabola in polars.