STEP III, 2006, Q5 MS

5	There are essentially two different configurations, corresponding to clockwise and anticlockwise arrangements of α, β, γ taken in order. In what follows, $\omega = \dfrac{-1+\sqrt{3}}{2}$, the cube root of unity with modulus 1 and argument $\dfrac{2\pi}{3}$; $1+\omega+\omega^2 = 0$ (*) is assumed.

	Then either $\beta-\gamma = \omega(\gamma-\alpha)$ and $\beta-\gamma = \omega^2(\gamma-\alpha)$ expresses equality of adjacent sides and the correct angle between them for each of the two cases; by SAS this establishes an equilateral triangle. These two are equivalent to $[\beta-\gamma-\omega(\gamma-\alpha)][\beta-\gamma-\omega^2(\gamma-\alpha)] = 0$. The required form is an expanded version of this, using (*). NB It is essential to be clear that the argument works both ways. If α, β, γ are the roots of the equation given, $-a = \alpha+\beta+\gamma, b = \alpha\beta+\beta\gamma+\gamma\alpha, c = -\alpha\beta\gamma$. Then $a^2-3b = \alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha$ so $a^2-3b=0$ is equivalent to the expression in the first part. Result follows. $z \to pw$ is an enlargement combined with rotation, so object and image are similar. $pw \to pw+q$ is a translation so object and image are congruent. Hence under the composition $z \to pw+q$ object and image are similar. Result follows. Aliter. Substitute $z = pw+q$ in the first equation, and simplify. Compare coefficients to determine A and B in terms of a, b and c. Then $a^2-3b=0 \Rightarrow A^2-3B=0$, so result follows.