STEP III, 2006, Q4

4 The function f satisfies the identity

$$\mathrm{f}(x) + \mathrm{f}(y) \equiv \mathrm{f}(x+y) \qquad (*)$$

for all x and y. Show that $2\mathrm{f}(x) \equiv \mathrm{f}(2x)$ and deduce that $\mathrm{f}''(0) = 0$. By considering the Maclaurin series for $\mathrm{f}(x)$, find the most general function that satisfies $(*)$.
[*Do not consider issues of existence or convergence of Maclaurin series in this question.*]

(i) By considering the function G, defined by $\ln\big(\mathrm{g}(x)\big) = \mathrm{G}(x)$, find the most general function that, for all x and y, satisfies the identity

$$\mathrm{g}(x)\mathrm{g}(y) \equiv \mathrm{g}(x+y)\,.$$

(ii) By considering the function H, defined by $\mathrm{h}(\mathrm{e}^u) = \mathrm{H}(u)$, find the most general function that satisfies, for all positive x and y, the identity

$$\mathrm{h}(x) + \mathrm{h}(y) \equiv \mathrm{h}(xy)\,.$$

(iii) Find the most general function t that, for all x and y, satisfies the identity

$$\mathrm{t}(x) + \mathrm{t}(y) \equiv \mathrm{t}(z)\,,$$

where $z = \dfrac{x+y}{1-xy}$.