STEP III, 2006, Q3

3 (i) Let

$$\tan x = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad \cot x = \frac{1}{x} + \sum_{n=0}^{\infty} b_n x^n$$

for $0 < x < \frac{1}{2}\pi$. Explain why $a_n = 0$ for even n.

Prove the identity

$$\cot x - \tan x \equiv 2 \cot 2x$$

and show that

$$a_n = (1 - 2^{n+1}) b_n .$$

(ii) Let $\operatorname{cosec} x = \dfrac{1}{x} + \displaystyle\sum_{n=0}^{\infty} c_n x^n$ for $0 < x < \frac{1}{2}\pi$. By considering $\cot x + \tan x$, or otherwise, show that

$$c_n = (2^{-n} - 1) b_n .$$

(iii) Show that

$$\left(1 + x \sum_{n=0}^{\infty} b_n x^n\right)^2 + x^2 = \left(1 + x \sum_{n=0}^{\infty} c_n x^n\right)^2 .$$

Deduce from this and the previous results that $a_1 = 1$, and find a_3.