STEP III, 2006, Q13 MS

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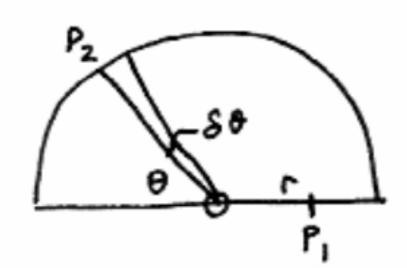

There are three cases to consider: (i) both on the circumference, (ii) P_1 on the diameter and P_2 on the circumference, and (iii) vice versa.

For case (i), if P_1 lies in the arc $(\alpha, \alpha+\delta\alpha)$ P_2 lies in the arc $(\theta, \theta+\delta\theta)$, with probability $\dfrac{\delta\theta}{\pi+2}$, the area is $\dfrac{1}{2}|r|\sin\theta$. The expected area given P_1 lies in the arc $(\alpha, \alpha+\delta\alpha)$ is by integration $\dfrac{1}{\pi+2}$.

For case (ii), if P_1 lies in $(r, r+\delta r)$ and P_2 lies in the arc $(\theta, \theta+\delta\theta)$, with probability $\dfrac{\delta\theta}{\pi+2}$, the area is $\dfrac{1}{2}|r|\sin\theta$. The expected area given P_1 lies in $(r, r+\delta r)$ from O is by integration $\dfrac{|r|}{\pi+2}$.

Case (iii) is essentially the same as case (ii).

Thus the expected area is

$$\int_0^{\pi} \frac{1}{\pi+2}.\frac{1}{\pi+2}\,d\alpha + 2\int_{-1}^{1} \frac{|r|}{\pi+2}.\frac{1}{\pi+2}\,dr$$

where the first integral corresponds to case (i) and the second to (ii) and (iii).

This evaluates to the answer given.