STEP III, 2006, Q10

10 A disc rotates freely in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is mk^2 (where $k > 0$). Along one diameter is a smooth narrow groove in which a particle of mass m slides freely. At time $t = 0$, the disc is rotating with angular speed Ω, and the particle is a distance a from the axis and is moving with speed V along the groove, towards the axis, where $k^2V^2 = \Omega^2a^2(k^2 + a^2)$.

Show that, at a later time t, while the particle is still moving towards the axis, the angular speed ω of the disc and the distance r of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \quad \text{and} \quad \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{\Omega^2 r^2 (k^2 + a^2)^2}{k^2(k^2 + r^2)}\,.$$

Deduce that

$$k\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r(k^2 + r^2)^{\frac{1}{2}}\,,$$

where θ is the angle through which the disc has turned by time t.

By making the substitution $u = k/r$, or otherwise, show that $r\sinh(\theta + \alpha) = k$, where $\sinh\alpha = k/a$. Deduce that the particle never reaches the axis.