STEP III, 2005, Q9 MS

9 Let the speeds of A and B after the first collision be u_1, u_2, then conservation of momentum gives

$$4eu_1 + (1-e)^2 u_2 = 4e(1-e)v - (1-e)^2 \cdot 2ev = 2ev(1-e^2)$$

and the restitution equation gives

$$u_2 - u_1 = (1-e)v + 2ev = e(1+e)v.$$

To find u_1, multiply the second equation by $(1-e)^2$ and subtract it from the first:

$$u_1 = \frac{2ev(1-e^2) - (1-e)^2 e(1+e)v}{4e + (1-e)^2} = \frac{ev(1-e^2)}{(1+e)^2}\left(2 - (1-e)\right) = e(1-e)v$$

To find u_2, multiply the second equation by $4e$ and add it to the first:

$$u_2 = \frac{2ev(1-e^2) + 4e^2(1+e)v}{4e + (1-e)^2} = \frac{2ev(1+e)}{(1+e)^2}\left((1-e) + 2e\right) = 2ev.$$

After B strikes the vertical wall, it rebounds with speed $2e^2v$, so if x is the distance from the wall at which the second collision occurs, the total time between collisions is

$$\frac{d-x}{e(1-e)v} = \frac{d}{2ev} + \frac{x}{2e^2v},$$

so that $2e(d-x) = (1-e)(ed+x)$ or $x = ed$.
Note that the situation is now that given initially, with all distances and speeds scaled by e. Thus the n^{th} collision occurs a distance de^{n-1} from the wall, and the speed of A between the n^{th} and the $(n+1)^{\text{th}}$ collisions is $(1-e)ve^n$, so the time between collisions is

$$\frac{de^{n-1} - de^n}{(1-e)ve^n} = \frac{d}{ev},$$

which is independent of n.