STEP III, 2005, Q8 MS

8 Direct use of the important result $|z|^2 = zz^*$ gives

$$|a-c|^2 = (a-c)(a^*-c^*) = aa^* + cc^* - ac^* - ca^*.$$

OAC is a right angle if and only if $|\text{AC}|^2 + |\text{OA}|^2 = |\text{OC}|^2$; that is, $|a-c|^2 + |a|^2 = |c|^2$ or, using the result above, $2aa^* - ac^* - ca^* = 0$.

The circle has centre C and radius AC, so complex numbers representing points on the circle satisfy $|z-c|^2 = |a-c|^2$ or $zz^* - zc^* - cz^* = aa^* - ac^* - ca^*$.
Because OA is a tangent to the circle, angle OAC is a right angle and so $2aa^* - ac^* - ca^* = 0$ as above; thus the condition for points to lie on the circle becomes $zz^* - zc^* - cz^* + aa^* = 0$.

P lies on this circle if and only if

$$aba^*b^* - abc^* - ca^*b^* + aa^* = 0$$

and P′ lies on the circle if and only if

$$\frac{aa^*}{bb^*} - \frac{a}{b^*}c^* - c\frac{a^*}{b} + aa^* = 0$$

but multiplying this by bb^* (which is not equal to zero) gives the same condition.

Conversely, if the points lie on the circle represented by $|z-c|^2 = |a-c|^2$,

$$aba^*b^* - abc^* - ca^*b^* + aa^* = 2aa^* - ac^* - ca^*$$

and

$$\frac{aa^*}{bb^*} - \frac{a}{b^*}c^* - c\frac{a^*}{b} = 2aa^* - ac^* - ca^*,$$

so that

$$aba^*b^* - abc^* - ca^*b^* + aa^* = bb^*(2aa^* - ac^* - ca^*)$$

and so, provided $bb^* \neq 1$, it must be the case that $2aa^* - ac^* - ca^* = 0$, and this shows that OAC is a right angle and hence that OA is a tangent to the circle.