STEP III, 2005, Q7 MS

7 Substituting $u = x^m$ gives

$$\int \frac{m\,\mathrm{d}x}{x\,\mathrm{f}(x^m)} = \int \frac{mx^{m-1}\mathrm{d}x}{x^m\,\mathrm{f}(x^m)} = \int \frac{\mathrm{d}u}{u\,\mathrm{f}(u)} = \mathrm{F}(u) + c = \mathrm{F}(x^m) + c$$

(i) $$\int \frac{\mathrm{d}x}{x^n - x} = \int \frac{\mathrm{d}x}{x\,(x^{n-1} - 1)},$$

so letting $u = x^{n-1}$ and $\mathrm{f}(u) = u - 1$,

$$\int \frac{(n-1)\mathrm{d}x}{x^n - x} = \int \frac{\mathrm{d}u}{u\,(u-1)} = \int \frac{1}{u-1} - \frac{1}{u}\,\mathrm{d}u = \ln\left|\frac{u-1}{u}\right|$$

so $$\int \frac{\mathrm{d}x}{x^n - x} = \frac{1}{n-1}\ln\left|\frac{x^{n-1} - 1}{x^{n-1}}\right| + c.$$

(ii) $$\int \frac{\mathrm{d}x}{\sqrt{x^n + x^2}} = \int \frac{\mathrm{d}x}{x\sqrt{x^{n-2} + 1}} \quad (\text{for } x > 0)$$

so letting $u = x^{n-2}$ and $\mathrm{f}(u) = \sqrt{u+1}$ (and assuming $n \neq 2$)

$$\int \frac{(n-2)\mathrm{d}x}{\sqrt{x^n + x^2}} = \int \frac{\mathrm{d}u}{u\,\sqrt{u+1}}.$$

Substituting $u = v^2 - 1$ with $v > 0$,

$$\int \frac{\mathrm{d}u}{u\,\sqrt{u+1}} = \int \frac{2v\mathrm{d}v}{(v^2-1)\,v} = \int \frac{1}{v-1} - \frac{1}{v+1}\,\mathrm{d}v = \ln\left|\frac{v-1}{v+1}\right|$$

so $$\int \frac{\mathrm{d}x}{\sqrt{x^n + x^2}} = \frac{1}{n-2}\ln\left|\frac{\sqrt{x^{n-2}+1} - 1}{\sqrt{x^{n-2}+1} + 1}\right| + c.$$