STEP III, 2005, Q6 MS

6 Direct substitution of $x = 2a\cosh\left(\frac{T}{3}\right)$ into the left hand side of the equation gives

$$\left(2a\cosh\left(\frac{T}{3}\right)\right)^3 - 6a^3\cosh\left(\frac{T}{3}\right) = 2a^3\left(4\left(\cosh\left(\frac{T}{3}\right)\right)^3 - 3\cosh\left(\frac{T}{3}\right)\right) = 2a^3\cosh T$$

(by the first result given at the start of the question).

Let $a^2 = b$, which is possible since $b > 0$, and $\cosh T = \frac{c}{a^3}$, which requires $\frac{c}{a^3} \geqslant 1$; but this holds if you choose a to have the same sign as c, since then $\frac{c}{a^3} > 0$ and $c^2 > b^3 = a^6$.

Then, by the second result given at the start of the question,

$$T = \ln\left(\frac{c}{a^3} + \sqrt{\frac{c^2}{a^6} - 1}\right) = \ln\left(\frac{c + \sqrt{c^2 - b^3}}{a^3}\right) = 3\ln\left(\frac{u}{a}\right),$$

so one of the roots of the equation $x^3 - 3bx = 2c$ is

$$2a\cosh\left(\ln\left(\frac{u}{a}\right)\right) = 2a\frac{\frac{u}{a} + \frac{a}{u}}{2} = u + \frac{b}{u}.$$

Note that, since $u + \frac{b}{u}$ is a root of the equation $x^3 - 3bx = 2c$,

$$2c = \left(u + \frac{b}{u}\right)^3 - 3b\left(u + \frac{b}{u}\right) = \left(u + \frac{b}{u}\right)\left(u^2 + \frac{b^2}{u^2} - b\right)$$

and that

$$u^2 + \frac{b^2}{u^2} - b - \left(u + \frac{b}{u}\right)^2 = -3b,$$

so

$$x^3 - 3bx - 2c = \left(x - \left(u + \frac{b}{u}\right)\right)\left(x^2 + \left(u + \frac{b}{u}\right)x + u^2 + \frac{b^2}{u^2} - b\right),$$

so the other two roots of $x^3 - 3bx = 2c$ are the roots of $x^2 + \left(u + \frac{b}{u}\right)x + u^2 + \frac{b^2}{u^2} - b = 0$, which are

$$\tfrac{1}{2}\left(-\left(u + \frac{b}{u}\right) \pm \sqrt{\left(u + \frac{b}{u}\right)^2 - 4\left(u^2 + \frac{b^2}{u^2} - b\right)}\right)$$

$$= \tfrac{1}{2}\left(-\left(u + \frac{b}{u}\right) \pm \sqrt{-3\left(u^2 + \frac{b^2}{u^2} - 2b\right)}\right) = \tfrac{1}{2}\left(-\left(u + \frac{b}{u}\right) \pm \sqrt{3}j\left(u - \frac{b}{u}\right)\right),$$

that is $\omega u + \omega^2\frac{b}{u}$ and $\omega^2 u + \omega\frac{b}{u}$.

In $x^3 - 6x = 6$, $b = 2$, $c = 3$, so $a = \sqrt{2}$ and so $u = \sqrt[3]{3+1} = 2^{\frac{2}{3}}$ and $\frac{b}{u} = 2^{\frac{1}{3}}$, so the solutions are $2^{\frac{1}{3}} + 2^{\frac{2}{3}}$, $\omega 2^{\frac{1}{3}} + \omega^2 2^{\frac{2}{3}}$ and $\omega^2 2^{\frac{1}{3}} + \omega 2^{\frac{2}{3}}$.