STEP III, 2005, Q6

6 In this question, you may use without proof the results

$$4\cosh^3 y - 3\cosh y = \cosh(3y) \quad \text{and} \quad \operatorname{arcosh} y = \ln\left(y + \sqrt{y^2-1}\right).$$

[**Note:** arcosh$y$ is another notation for $\cosh^{-1} y$]

Show that the equation $x^3 - 3a^2x = 2a^3\cosh T$ is satisfied by $2a\cosh\left(\frac{1}{3}T\right)$ and hence that, if $c^2 \geqslant b^3 > 0$, one of the roots of the equation $x^3 - 3bx = 2c$ is $u + \dfrac{b}{u}$, where $u = \left(c + \sqrt{c^2-b^3}\right)^{\frac{1}{3}}$.

Show that the other two roots of the equation $x^3 - 3bx = 2c$ are the roots of the quadratic equation $x^2 + (u + \dfrac{b}{u})x + u^2 + \dfrac{b^2}{u^2} - b = 0$, and find these roots in terms of u, b and ω, where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$.

Solve completely the equation $x^3 - 6x = 6$.