STEP III, 2005, Q5 MS

5 The point on the curve with the required gradient is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b = m \text{ or } x = \frac{m-b}{2a},$$

with

$$y = a\left(\frac{m-b}{2a}\right)^2 + b\left(\frac{m-b}{2a}\right) + c = \frac{m^2-b^2}{4a} + c.$$

The equation of the tangent is therefore:

$$y - mx = a\left(\frac{m-b}{2a}\right)^2 + b\left(\frac{m-b}{2a}\right) + c - m\left(\frac{m-b}{2a}\right)$$

$$= c - \frac{(m-b)}{2a}\left(m - b - \frac{(m-b)}{2}\right) = c - \frac{(m-b)^2}{4a}.$$

The curves have a common tangent with gradient m if and only if the equations of the tangents to the two curves with gradient m are identical; that is, have the same intercept, so if and only if

$$c_1 - \frac{(m-b_1)^2}{4a_1} = c_2 - \frac{(m-b_2)^2}{4a_2}$$

that is,

$$4a_1a_2c_1 - a_2m^2 + 2a_2b_1m - a_2b_1^2 = 4a_1a_2c_2 - a_1m^2 + 2a_1b_2m - a_1b_2^2$$

which gives the quoted result.

There is exactly one common tangent when $a_1 \neq a_2$ when the quadratic equation for m has exactly one root, which occurs if and only if the discriminant of the equation is zero; that is

$$4(a_1b_2 - a_2b_1)^2 = 4(a_2 - a_1)(4a_1a_2(c_2 - c_1) + a_2b_1^2 - a_1b_2^2)$$

$$\Leftrightarrow \quad 4a_1^2b_2^2 - 8a_1a_2b_1b_2 + 4a_2^2b_1^2 = 16a_1a_2(a_2 - a_1)(c_2 - c_1) + 4a_2^2b_1^2 + 4a_1^2b_2^2 - 4a_1a_2(b_1^2 + b_2^2)$$

$$\Leftrightarrow \quad 4a_1a_2(b_1^2 + b_2^2) - 8a_1a_2b_1b_2 = 16a_1a_2(a_2 - a_1)(c_2 - c_1)$$

$$\Leftrightarrow \quad b_1^2 + b_2^2 - 2b_1b_2 = 4(a_2 - a_1)(c_2 - c_1) \quad \text{(dividing by } 4a_1a_2 \text{ which is non-zero).}$$

The curves touch if there is exactly one solution to the simultaneous equations

$$y = a_1x^2 + b_1x + c_1 \text{ and } y = a_2x^2 + b_2x + c_2;$$

that is, if the equation $(a_2 - a_1)x^2 + (b_2 - b_1)x + (c_2 - c_1) = 0$ has exactly one root so, again using the discriminant condition, if and only if $(b_2 - b_1)^2 = 4(a_2 - a_1)(c_2 - c_1)$, which is the same condition.

If $a_1 = a_2$ the curves have exactly one common tangent if there is exactly one solution to

$$2m(b_2 - b_1) + 4a(c_2 - c_1) + (b_1^2 - b_2^2) = 0;$$

since this is just a linear equation, the only condition is that $b_2 - b_1 \neq 0$.