STEP III, 2005, Q5

5 Let P be the point on the curve $y = ax^2 + bx + c$ (where a is non-zero) at which the gradient is m. Show that the equation of the tangent at P is

$$y - mx = c - \frac{(m-b)^2}{4a} .$$

Show that the curves $y = a_1x^2 + b_1x + c_1$ and $y = a_2x^2 + b_2x + c_2$ (where a_1 and a_2 are non-zero) have a common tangent with gradient m if and only if

$$(a_2 - a_1)m^2 + 2(a_1b_2 - a_2b_1)m + 4a_1a_2(c_2 - c_1) + a_2b_1^2 - a_1b_2^2 = 0 .$$

Show that, in the case $a_1 \neq a_2$, the two curves have exactly one common tangent if and only if they touch each other. In the case $a_1 = a_2$, find a necessary and sufficient condition for the two curves to have exactly one common tangent.