STEP III, 2005, Q4 MS

4 For the base case you need to verify that $u_{2n} = \frac{b}{a}u_{2n-1}$ and $u_{2n+1} = cu_{2n}$ when $n = 1$:

$u_1 = a$, $u_2 = b$ so $u_{2n} = \frac{b}{a}u_{2n-1}$ when $n = 1$;

$u_3 = \frac{u_2}{u_1}(ku_1 - u_2) = u_2\frac{ka - b}{a}$ so $u_{2n+1} = cu_{2n}$ when $n = 1$, provided $c = k - \frac{b}{a}$.

For the induction step, assume that $u_{2n} = \frac{b}{a}u_{2n-1}$ and $u_{2n+1} = cu_{2n}$ when $n = N$ then

$$u_{2N+2} = \frac{u_{2N+1}}{u_{2N}}(ku_{2N} - u_{2N+1})$$

$$= u_{2N+1}(k - c) \text{ (by the induction hypothesis)}$$

$$= \frac{b}{a}u_{2N+1} \text{ (by the definition of } c\text{)}$$

and

$$u_{2N+3} = \frac{u_{2N+2}}{u_{2N+1}}(ku_{2N+1} - u_{2N+2})$$

$$= u_{2N+2}\left(k - \frac{b}{a}\right) \text{ (by what has just been shown)}$$

$$= cu_{2N+2}.$$

which completes the induction.

Hence $u_{2n} = \frac{bc}{a}u_{2n-2} = \ldots = \left(\frac{bc}{a}\right)^{n-1} u_2 = b\left(\frac{bc}{a}\right)^{n-1}$

and $u_{2n-1} = \frac{bc}{a}u_{2n-3} = \ldots = \left(\frac{bc}{a}\right)^{n-1} u_1 = a\left(\frac{bc}{a}\right)^{n-1}$

(i) For u_n to be geometric requires $\frac{u_{2n}}{u_{2n-1}} = \frac{u_{2n+1}}{u_{2n}}$; that is, $\frac{b}{a} = c = k - \frac{b}{a}$ or $ak = 2b$;

(ii) For u_n to have period 2 requires $u_{2n+1} = u_{2n-1}$, but $u_{2n+1} = cu_{2n} = \frac{cb}{a}u_{2n-1}$, so it is necessary that $\frac{cb}{a} = 1$ or $a^2 + b^2 = kab$;

(iii) For u_n to have period 4 requires $u_{2n+3} = u_{2n-1}$ so, by the previous part, it is necessary that $\left(\frac{cb}{a}\right)^2 = 1$ but $\frac{cb}{a} \neq 1$ (to avoid period 2) so $\frac{cb}{a} = -1$ or $b^2 - a^2 = kab$.