

STEP III, 2005, Q4 MS

- 4 For the base case you need to verify that $u_{2n} = \frac{b}{a}u_{2n-1}$ and $u_{2n+1} = cu_{2n}$ when $n = 1$:

$$u_1 = a, u_2 = b \text{ so } u_{2n} = \frac{b}{a}u_{2n-1} \text{ when } n = 1;$$

$$u_3 = \frac{u_2}{u_1}(ku_1 - u_2) = u_2 \frac{ka - b}{a} \text{ so } u_{2n+1} = cu_{2n} \text{ when } n = 1, \text{ provided } c = k - \frac{b}{a}.$$

For the induction step, assume that $u_{2n} = \frac{b}{a}u_{2n-1}$ and $u_{2n+1} = cu_{2n}$ when $n = N$ then

$$\begin{aligned} u_{2N+2} &= \frac{u_{2N+1}}{u_{2N}}(ku_{2N} - u_{2N+1}) \\ &= u_{2N+1}(k - c) \text{ (by the induction hypothesis)} \\ &= \frac{b}{a}u_{2N+1} \text{ (by the definition of } c) \end{aligned}$$

and

$$\begin{aligned} u_{2N+3} &= \frac{u_{2N+2}}{u_{2N+1}}(ku_{2N+1} - u_{2N+2}) \\ &= u_{2N+2} \left(k - \frac{b}{a} \right) \text{ (by what has just been shown)} \\ &= cu_{2N+2}. \end{aligned}$$

which completes the induction.

$$\text{Hence } u_{2n} = \frac{bc}{a}u_{2n-2} = \dots = \left(\frac{bc}{a}\right)^{n-1} u_2 = b \left(\frac{bc}{a}\right)^{n-1}$$

$$\text{and } u_{2n-1} = \frac{bc}{a}u_{2n-3} = \dots = \left(\frac{bc}{a}\right)^{n-1} u_1 = a \left(\frac{bc}{a}\right)^{n-1}$$

- (i) For u_n to be geometric requires $\frac{u_{2n}}{u_{2n-1}} = \frac{u_{2n+1}}{u_{2n}}$; that is, $\frac{b}{a} = c = k - \frac{b}{a}$ or $ak = 2b$;
- (ii) For u_n to have period 2 requires $u_{2n+1} = u_{2n-1}$, but $u_{2n+1} = cu_{2n} = \frac{cb}{a}u_{2n-1}$, so it is necessary that $\frac{cb}{a} = 1$ or $a^2 + b^2 = kab$;
- (iii) For u_n to have period 4 requires $u_{2n+3} = u_{2n-1}$ so, by the previous part, it is necessary that $\left(\frac{cb}{a}\right)^2 = 1$ but $\frac{cb}{a} \neq 1$ (to avoid period 2) so $\frac{cb}{a} = -1$ or $b^2 - a^2 = kab$.



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