STEP III, 2005, Q4

4 The sequence u_n $(n = 1, 2, \ldots)$ satisfies the recurrence relation

$$u_{n+2} = \frac{u_{n+1}}{u_n}(ku_n - u_{n+1})$$

where k is a constant.

If $u_1 = a$ and $u_2 = b$, where a and b are non-zero and $b \neq ka$, prove by induction that

$$u_{2n} = \left(\frac{b}{a}\right)u_{2n-1}$$

$$u_{2n+1} = cu_{2n}$$

for $n \geqslant 1$, where c is a constant to be found in terms of k, a and b. Hence express u_{2n} and u_{2n-1} in terms of a, b, c and n.

Find conditions on a, b and k in the three cases:

(i) the sequence u_n is geometric;

(ii) the sequence u_n has period 2;

(iii) the sequence u_n has period 4.