STEP III, 2005, Q3 MS

3 Direct substitution gives

$$f(g(x)) = (x^2 + rx + s)^2 + p(x^2 + rx + s) + q$$

$$= x^4 + 2rx^3 + (r^2 + 2s + p)x^2 + (2rs + rp)x + s^2 + ps + q.$$

If $x^4 + ax^3 + bx^2 + cx + d$ is to have this form then it is necessary to choose $r = \frac{a}{2}$ and to choose s and p to satisfy $2s + p = b - r^2 = b - \frac{a^2}{4}$ and $r(2s + p) = c$ or $a(2s + p) = 2c$. Thus $a\left(b - \frac{a^2}{4}\right) = 2c$ is a necessary condition for this to be possible.

It is also sufficient: in fact, pick $p = 0$; then $s = \frac{4b - a^2}{8}$ and $q = d - s^2$ will do.

Expanding the second form gives

$$(x^2 + vx + w)^2 - k = x^4 + 2vx^3 + (v^2 + 2w)x^2 + 2vwx + w^2 - k,$$

but this is identical to $x^4 + 2rx^3 + (r^2 + 2s + p)x^2 + (2rs + rp)x + s^2 + ps + q$ with $p = 0$, $v = r$, $w = s$ and $k = -q$, and so, since the sufficiency demonstrated above allowed the choice $p = 0$, the condition is the same.

To solve the final equation, write the quartic in the second form:

$$x^4 - 4x^3 + 10x^2 - 12x + 4 = (x^2 - 2x + 3)^2 - 5 = 0$$

so

$$x^2 - 2x + 3 - \sqrt{5} = 0 \text{ or } x^2 - 2x + 3 + \sqrt{5} = 0$$

so

$$x = 1 \pm \sqrt{\sqrt{5} - 2} \text{ or } 1 \pm j\sqrt{\sqrt{5} + 2}.$$