STEP III, 2005, Q3

3 Let $\mathrm{f}(x) = x^2 + px + q$ and $\mathrm{g}(x) = x^2 + rx + s$. Find an expression for $\mathrm{f}(\mathrm{g}(x))$ and hence find a necessary and sufficient condition on a, b and c for it to be possible to write the quartic expression $x^4 + ax^3 + bx^2 + cx + d$ in the form $\mathrm{f}(\mathrm{g}(x))$, for some choice of values of p, q, r and s.

Show further that this condition holds if and only if it is possible to write the quartic expression $x^4 + ax^3 + bx^2 + cx + d$ in the form $(x^2 + vx + w)^2 - k$, for some choice of values of v, w and k.

Find the roots of the quartic equation $x^4 - 4x^3 + 10x^2 - 12x + 4 = 0$.