

## STEP III, 2005, Q2 MS

- 2 This equation can be solved by separating the variables:

$$\int \frac{2 dy}{y} = - \int \frac{2x dx}{x^2 + a^2} \quad \text{so} \quad \ln(y^2) = -\ln(x^2 + a^2) + k \quad \text{or} \quad y^2(x^2 + a^2) = c^2.$$

The curve has two branches: one has  $y > 0$ , reflection symmetry about the  $y$ -axis, a maximum at  $(0, \frac{c}{a})$  and  $y \rightarrow 0$  as  $|x| \rightarrow \infty$ ; the other has  $y < 0$  and is a reflection of the first branch in the  $x$ -axis.

$$\frac{d}{dx}(x^2 + y^2) = 2x - 2y \frac{dy}{dx} = 2x - \frac{2xy^2}{x^2 + a^2} = 2x - \frac{2xc^2}{(x^2 + a^2)^2}$$

$$\frac{d^2}{dx^2}(x^2 + y^2) = 2 - \frac{2c^2}{(x^2 + a^2)^2} + \frac{4xc^2 \cdot 2x}{(x^2 + a^2)^3} = 2 \left( 1 - \frac{c^2}{(x^2 + a^2)^2} \right) + \frac{8c^2x^2}{(x^2 + a^2)^3}$$

- (i)  $\frac{d}{dx}(x^2 + y^2) = 0$  when  $x = 0$  and when  $c^2 = (x^2 + a^2)^2$ , but the latter is not possible if  $0 < c < a^2$ . If  $x = 0$ ,  $y = \pm \frac{c}{a}$  and  $\frac{d^2}{dx^2}(x^2 + y^2) = 1 - \frac{c^2}{a^4}$  which is positive if  $0 < c < a^2$ , indicating a local minimum. Hence the points on the curve whose distance from the origin is least are  $(0, \pm \frac{c}{a})$ .
- (ii) If  $c > a^2$  then  $\frac{d^2}{dx^2}(x^2 + y^2)$  is negative at  $x = 0$ , indicating a local maximum there; but in this case there are further stationary points at  $x^2 = c - a^2$ ,  $y = \pm \sqrt{c}$  and at these points  $\frac{d^2}{dx^2}(x^2 + y^2) = \frac{8x^2}{c} > 0$ . Hence the points on the curve whose distance from the origin is least are  $(\pm \sqrt{c - a^2}, \pm \sqrt{c})$ .



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