STEP III, 2005, Q2 MS

2 This equation can be solved by separating the variables:

$$\int \frac{2\,\mathrm{d}y}{y} = -\int \frac{2x\,\mathrm{d}x}{x^2+a^2} \quad \text{so} \quad \ln(y^2) = -\ln(x^2+a^2)+k \quad \text{or} \quad y^2(x^2+a^2)=c^2.$$

The curve has two branches: one has $y>0$, reflection symmetry about the y-axis, a maximum at $\left(0\,,\ \frac{c}{a}\right)$ and $y \longrightarrow 0$ as $|x| \longrightarrow \infty$; the other has $y<0$ and is a reflection of the first branch in the x-axis.

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^2) = 2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{2xy^2}{x^2+a^2} = 2x - \frac{2xc^2}{(x^2+a^2)^2}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2) = 2 - \frac{2c^2}{(x^2+a^2)^2} + \frac{4xc^2\cdot 2x}{(x^2+a^2)^3} = 2\left(1-\frac{c^2}{(x^2+a^2)^2}\right) + \frac{8c^2x^2}{(x^2+a^2)^3}$$

(i) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^2)=0$ when $x=0$ and when $c^2=(x^2+a^2)^2$, but the latter is not possible if $0<c<a^2$. If $x=0$, $y=\pm\frac{c}{a}$ and $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2) = 1-\frac{c^2}{a^4}$ which is positive if $0<c<a^2$, indicating a local minimum. Hence the points on the curve whose distance from the origin is least are $\left(0\,,\ \pm\frac{c}{a}\right)$.

(ii) If $c>a^2$ then $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2)$ is negative at $x=0$, indicating a local maximum there; but in this case there are further stationary points at $x^2=c-a^2\,,\ y=\pm\sqrt{c}$ and at these points $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2) = \frac{8x^2}{c} > 0$. Hence the points on the curve whose distance from the origin is least are $(\pm\sqrt{c-a^2}\,,\ \pm\sqrt{c})$.