STEP III, 2005, Q1 MS

1 To prove the first part, use the results: $\cos B = \sin\left(\frac{\pi}{2} - B\right)$, whatever the value of B; and

$$\sin A = \sin Y \Leftrightarrow A = Y + 2n\pi \text{ or } A = \pi - Y + 2n\pi;$$

thus here, replacing Y by $\frac{\pi}{2} - B$, $A = 2n\pi + \frac{\pi}{2} \pm B$.

For the next part, it is probably easiest to use the fact that $a\sin x \pm b\cos x$ can be written in the form $R\sin(x \pm \alpha)$; here,

$$\sin x \pm \cos x = \sqrt{2}\left(\sin x \cos\frac{\pi}{4} \pm \cos x \sin\frac{\pi}{4}\right) = \sqrt{2}\sin\left(x \pm \frac{\pi}{4}\right)$$

so $|\sin x \pm \cos x| \leqslant \sqrt{2}$.

Now, from the first part,

$$\sin(\sin x) = \cos(\cos x) \Leftrightarrow \sin x = 2n\pi + \frac{\pi}{2} \pm \cos x$$

so

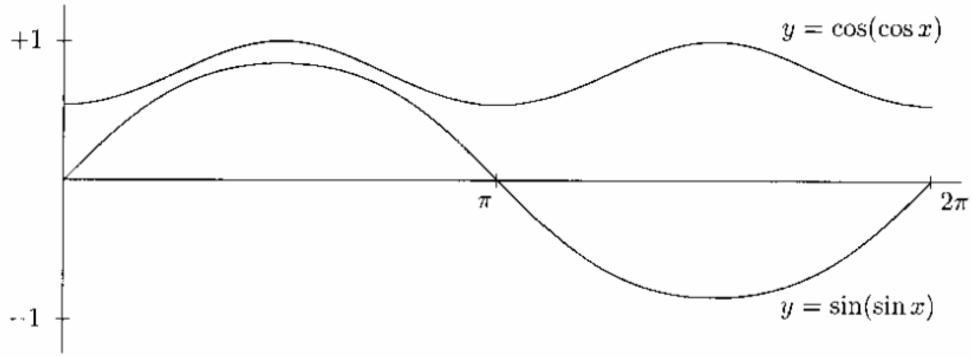
$$|\sin x \pm \cos x| \geqslant \left|2n\pi + \frac{\pi}{2}\right| \geqslant \frac{\pi}{2} > \sqrt{2},$$

which is a contradiction.

All the curves asked for have period 2π, so they will be sketched for x in this range only.

For $y = \sin(\sin x)$, $y = 0$ when $\sin x = 0$ only (since $|\sin x| < \pi$), so at 0, π and 2π; the turning points are at $\cos x \cos(\sin x) = 0$, so when $\cos x = 0$, that is at $x = \frac{\pi}{2}, \frac{3\pi}{2}$, or when $\cos(\sin x) = 0$,which is impossible since $|\sin x| < \frac{\pi}{2}$; the turning points are a maximum at $\left(\frac{\pi}{2}, \sin(1)\right)$ and a minimum at $\left(\frac{3\pi}{2}, -\sin(1)\right)$, where $\sin(1) \approx 0{\cdot}84$.

For $y = \cos(\cos x)$, $y > 0$ for all x, since $|\cos x| \leqslant 1 < \frac{\pi}{2}$; the turning points are at $\sin x \sin(\cos x) = 0$, so when either $\sin x = 0$ or $\cos x = 0$, that is at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$; the turning points are maxima at $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, 1\right)$, and minima at $(0, \cos(1))$, $(\pi, \cos(1))$, $(2\pi, \cos(1))$, where $\cos(1) \approx 0{\cdot}54$.



For the curve $y = \sin(2\sin x)$, $y = 0$ if $2\sin x$ is a multiple of π, which is only possible if $\sin x = 0$ (since $|2\sin x| < \pi$), so when x is 0, π and 2π; the turning points are at $2\cos x\cos(2\sin x) = 0$; so when $\cos x = 0$, that is at $x = \frac{\pi}{2}, \frac{3\pi}{2}$, or when $2\sin x$ is an odd multiple of $\frac{\pi}{2}$, which is only possible if $2\sin x = \pm\frac{\pi}{2}$, so when $\sin x = \frac{\pi}{4} \approx \pm 0{\cdot}8$; the turning points are a minimum at $\left(\frac{\pi}{2}, \sin 2\right)$, where $\sin 2 \approx 0{\cdot}91$ and maxima either side of this, with y-coordinates 1 and a maximum at $\left(\frac{3\pi}{2}, -\sin 2\right)$ with minima either side with y-coordinates -1.