STEP III, 2005, Q14 MS

14 The integral of the density function from 0 to infinity must equal 1:

$$1 = \int_0^\infty \frac{Ck^{a+1}x^a}{(x+k)^{2a+2}}\,dx = Ck^{a+1}\frac{a!(2a-a)!}{(2a+1)!k^{2a-a+1}},$$

using the given result with $m = a$ and $n = 2a$

$$= C\frac{a!a!}{(2a+1)!} \text{ so } C = \frac{(2a+1)!}{a!a!}.$$

Use the substitution $x = \dfrac{k^2}{u}$:

$$\int_0^v \frac{x^a}{(x+k)^{2a+2}}\,dx = \int_\infty^{\frac{k^2}{v}} \frac{k^{2a}}{u^a\left(\frac{k}{u}(u+k)\right)^{2a+2}}\frac{-k^2du}{u^2} = \int_{\frac{k^2}{v}}^\infty \frac{u^a}{(u+k)^{2a+2}}\,du$$

Choosing $v = k$ gives $\displaystyle\int_0^k f(x)\,dx = \int_k^\infty f(x)\,dx$, so k is the median, and

$$E[V] = \int_0^\infty \frac{C\,k^{a+1}\,x^{a+1}}{(x+k)^{2a+2}}\,dx = \frac{(2a+1)!}{a!\,a!}k^{a+1}\frac{(a+1)!\,(a-1)!}{(2a+1)!\,k^a} = k\left(\frac{a+1}{a}\right).$$

Notice that $T < t$ if and only if $V > \dfrac{s}{t}$, so that

$$P(T < t) = P\left(V > \frac{s}{t}\right) = \int_{\frac{s}{t}}^\infty \frac{C\,k^{a+1}\,x^a}{(x+k)^{2a+2}}\,dx$$

and making the substitution $x = \dfrac{s}{u}$:

$$= \int_t^0 \frac{C\,k^{a+1}\,s^a}{u^a\left(\frac{k}{u}\left(\frac{s}{k}+u\right)\right)^{2a+2}}\frac{-s\,du}{u^2} = \int_0^t \frac{C\,u^a\left(\frac{s}{k}\right)^{a+1}}{\left(\frac{s}{k}+u\right)^{2a+2}}\,du$$

so the density function is $\dfrac{C\,u^a\left(\frac{s}{k}\right)^{a+1}}{\left(\frac{s}{k}+u\right)^{2a+2}}$, which is the same as that of V with k replaced by $\dfrac{s}{k}$.

This means that the median time is $\dfrac{s}{k}$ and that the expected time is $\dfrac{s}{k}\left(\dfrac{a+1}{a}\right)$ and hence

median velocity $\times$ median time $= s$, but $E[V]\,E[T] = s\left(\dfrac{a+1}{a}\right)^2$, which is greater than s.