

## STEP III, 2005, Q14 MS

14 The integral of the density function from 0 to infinity must equal 1:

$$1 = \int_0^{\infty} \frac{Ck^{a+1}x^a}{(x+k)^{2a+2}} dx = Ck^{a+1} \frac{a!(2a-a)!}{(2a+1)!k^{2a-a+1}},$$

using the given result with  $m = a$  and  $n = 2a$

$$= C \frac{a!a!}{(2a+1)!} \text{ so } C = \frac{(2a+1)!}{a!a!}.$$

Use the substitution  $x = \frac{k^2}{u}$ :

$$\int_0^v \frac{x^a}{(x+k)^{2a+2}} dx = \int_{\frac{k^2}{v}}^{\frac{k^2}{k}} \frac{k^{2a}}{u^a \left(\frac{k}{u}(u+k)\right)^{2a+2}} \frac{-k^2 du}{u^2} = \int_{\frac{k^2}{v}}^{\infty} \frac{u^a}{(u+k)^{2a+2}} du$$

Choosing  $v = k$  gives  $\int_0^k f(x) dx = \int_k^{\infty} f(x) dx$ , so  $k$  is the median, and

$$E[V] = \int_0^{\infty} \frac{Ck^{a+1}x^{a+1}}{(x+k)^{2a+2}} dx = \frac{(2a+1)!}{a!a!} k^{a+1} \frac{(a+1)!(a-1)!}{(2a+1)!k^a} = k \left(\frac{a+1}{a}\right).$$

Notice that  $T < t$  if and only if  $V > \frac{s}{t}$ , so that

$$P(T < t) = P\left(V > \frac{s}{t}\right) = \int_{\frac{s}{t}}^{\infty} \frac{Ck^{a+1}x^a}{(x+k)^{2a+2}} dx$$

and making the substitution  $x = \frac{s}{u}$ :

$$= \int_t^0 \frac{Ck^{a+1}s^a}{u^a \left(\frac{k}{u}\left(\frac{s}{k} + u\right)\right)^{2a+2}} \frac{-s du}{u^2} = \int_0^t \frac{C u^a \left(\frac{s}{k}\right)^{a+1}}{\left(\frac{s}{k} + u\right)^{2a+2}} du$$

so the density function is  $\frac{C u^a \left(\frac{s}{k}\right)^{a+1}}{\left(\frac{s}{k} + u\right)^{2a+2}}$ , which is the same as that of  $V$  with  $k$  replaced by  $\frac{s}{k}$ .

This means that the median time is  $\frac{s}{k}$  and that the expected time is  $\frac{s}{k} \left(\frac{a+1}{a}\right)$  and hence median velocity  $\times$  median time =  $s$ , but  $E[V]E[T] = s \left(\frac{a+1}{a}\right)^2$ , which is greater than  $s$ .



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