STEP III, 2005, Q14

14 In this question, you may use the result

$$\int_0^\infty \frac{t^m}{(t+k)^{n+2}}\,\mathrm{d}t = \frac{m!\,(n-m)!}{(n+1)!\,k^{n-m+1}}\,,$$

where m and n are positive integers with $n \geqslant m$, and where $k > 0$.

The random variable V has density function

$$\mathrm{f}(x) = \frac{C\,k^{a+1}\,x^a}{(x+k)^{2a+2}} \qquad (0 \leqslant x < \infty)\,,$$

where a is a positive integer. Show that $C = \dfrac{(2a+1)!}{a!\,a!}$.

Show, by means of a suitable substitution, that

$$\int_0^v \frac{x^a}{(x+k)^{2a+2}}\,\mathrm{d}x = \int_{\frac{k^2}{v}}^\infty \frac{u^a}{(u+k)^{2a+2}}\,\mathrm{d}u$$

and deduce that the median value of V is k. Find the expected value of V.

The random variable V represents the speed of a randomly chosen gas molecule. The time taken for such a particle to travel a fixed distance s is given by the random variable $T = \dfrac{s}{V}$.

Show that

$$\mathrm{P}(T < t) = \int_{\frac{s}{t}}^\infty \frac{C\,k^{a+1}\,x^a}{(x+k)^{2a+2}}\,\mathrm{d}x \qquad (*)$$

and hence find the density function of T. You may find it helpful to make the substitution $u = \dfrac{s}{x}$ in the integral $(*)$.

Hence show that the product of the median time and the median speed is equal to the distance s, but that the product of the expected time and the expected speed is greater than s.