STEP III, 2005, Q13 MS

13 The probability that the player wins exactly £3 is equal to the probability that the next 3 scores which lie in the range 0 to w are non zero, and the fourth score which lies in the range 0 to w is zero as the occurrence of outcomes which lead to the game continuing does not affect the final result.

Hence the probability that the player wins exactly £3 is equal to $\left(\dfrac{w}{w+1}\right)^3 \dfrac{1}{w+1}$.

Similarly, the probability that he wins exactly £r is $\left(\dfrac{w}{w+1}\right)^r \dfrac{1}{w+1}$ and so his expected winnings are

$$\sum_{r=0}^{\infty} r\left(\frac{w}{w+1}\right)^r \frac{1}{w+1} = \frac{w}{(w+1)^2}\sum_{r=1}^{\infty} r\left(\frac{w}{w+1}\right)^{r-1} = \frac{w}{(w+1)^2}\frac{1}{\left(1-\dfrac{w}{w+1}\right)^2} = w.$$

This calculation uses the result $\sum_{r=1}^{\infty} rp^{r-1} = \sum_{r=0}^{\infty} rp^{r-1} = \dfrac{1}{(1-p)^2}$, which you may know, or can be derived by noticing that $\sum_{r=0}^{\infty} rp^{r-1} = \dfrac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{r=0}^{\infty} p^r\right)$ and that $\sum_{r=0}^{\infty} p^r = \dfrac{1}{1-p}$, recognising an infinite geometric series.

In a second game, consider the cards set out in the order in which they will be drawn. Then only $w+1$ cards are relevant, and the zero card is equally likely to be any of these, so that the probability that he wins exactly £r is $\dfrac{1}{w+1}$ (for $r = 1, 2, \ldots w$) and so his expected winnings are now

$$\sum_{r=0}^{w} r\frac{1}{w+1} = \frac{1}{w+1}\sum_{r=0}^{w} r = \frac{1}{w+1}\tfrac{1}{2}w(w+1) = \tfrac{1}{2}w.$$