STEP III, 2005, Q12 MS

12 If $X = aT + b(T_1 + T_2 + T_3 + T_4)$ then

$$\mathrm{E}[X] = at + b(t_1 + t_2 + t_3 + t_4) = (a + b)t,$$

since $t_1 + t_2 + t_3 + t_4 = t$, by definition. You require $\mathrm{E}[X] = t$ which gives $a + b = 1$.

$$\mathrm{Var}[X] = a^2\mathrm{Var}[T] + b^2\mathrm{Var}[T_1] + b^2\mathrm{Var}[T_2] + b^2\mathrm{Var}[T_3] + b^2\mathrm{Var}[T_4],$$

assuming the errors made by the five timers are independent,

$$= (a^2 + 4b^2)\sigma^2 = (a^2 + 4(1-a)^2)\sigma^2 = (5a^2 - 8a + 4)\sigma^2 = \left(5\left(a - \frac{4}{5}\right)^2 + \frac{4}{5}\right)\sigma^2$$

which has a minimum value of $\frac{4}{5}\sigma^2$ when $a = \frac{4}{5}$; that is, when $X = \frac{1}{5}(4T + (T_1 + T_2 + T_3 + T_4))$.

Rearranging the identity $\mathrm{Var}[Y] = \mathrm{E}[Y^2] - \mathrm{E}[Y]^2$ gives $\mathrm{E}[Y^2] = \mathrm{Var}[Y] + \mathrm{E}[Y]^2$, so if $Y = cT + d(T_1 + T_2 + T_3 + T_4)$ then

$$\mathrm{E}[Y^2] = (c^2 + 4d^2)\sigma^2 + (ct + d(t_1 + t_2 + t_3 + t_4))^2 = (c^2 + 4d^2)\sigma^2 + (c + d)^2t^2,$$

which is equal to σ^2 regardless of the true lap times if $c + d = 0$ and $1 = c^2 + 4d^2 = 5c^2$, so that $c = \dfrac{1}{\sqrt{5}}$ and $Y^2 = \frac{1}{5}(T - (T_1 + T_2 + T_3 + T_4))^2$.

The timers could reasonably expect the true time for the race to lie within k estimated standard errors of the estimated value where, for instance, $k = 2$ or 3; so between

$$\tfrac{1}{5}(4T + (T_1 + T_2 + T_3 + T_4)) + k\sqrt{\tfrac{4}{5} \times \tfrac{1}{5}(T - (T_1 + T_2 + T_3 + T_4))^2}$$

and

$$\tfrac{1}{5}(4T + (T_1 + T_2 + T_3 + T_4)) - k\sqrt{\tfrac{4}{5} \times \tfrac{1}{5}(T - (T_1 + T_2 + T_3 + T_4))^2};$$

that is, between $220.1 + \dfrac{k}{5}$ and $220.1 - \dfrac{k}{5}$. For $k = 2$, this is the interval $[219{\cdot}7\,,\ 220{\cdot}5]$.