STEP III, 2005, Q11 MS

11 The energy of the system is the sum of the gravitational potential energy (GPE) and the kinetic energy (KE), with KE $= \frac{1}{2}m\left(a^2+b^2+c^2\right)\dot{\theta}^2$ and, taking the zero of GPE to be at the height of the spindle,

$$\begin{aligned}\text{GPE} &= mg\left(a\cos\theta - b\cos\left(\frac{\pi}{3}-\theta\right) - c\cos\left(\frac{\pi}{3}+\theta\right)\right)\\ &= \tfrac{1}{2}mg\left((2a-b-c)\cos\theta - (b-c)\sqrt{3}\sin\theta\right)\end{aligned}$$

simplifying using the $\cos(A\pm B)$ identities and the exact values of $\sin\frac{\pi}{3}$ and $\cos\frac{\pi}{3}$.

Equilibrium occurs at a stationary point of the GPE:

$$\frac{\text{dGPE}}{\text{d}\theta} = \tfrac{1}{2}mg\left(-(2a-b-c)\sin\theta - (b-c)\sqrt{3}\cos\theta\right) = 0$$

or when the total moment about the spindle of the gravitational forces on the particles is zero:

$$mg\left(a\sin\theta + b\sin\left(\frac{\pi}{3}-\theta\right) - c\sin\left(\frac{\pi}{3}+\theta\right)\right) = 0$$

which simplifies to the same equation using the $\sin(A\pm B)$ identities.

This equation is satisfied if

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = -\frac{(b-c)\sqrt{3}}{2a-b-c} < 0$$

which has two solutions between 0 and 2π unless $a=b=c=0$. It is useful to realise here that $\tan\theta = -\frac{p}{q} \Rightarrow \sin\theta = \mp\frac{p}{h}$ and $\cos\theta = \pm\frac{q}{h}$, where h is the positive number with $h^2 = p^2+q^2$.

In this case, $\sin\theta == -\frac{(b-c)\sqrt{3}}{2R}$ and $\cos\theta = \frac{(2a-b-c)}{2R}$ give one equilibrium and $\sin\theta = \frac{(b-c)\sqrt{3}}{2R}$ and $\cos\theta = -\frac{(2a-b-c)}{2R}$ the other, where

$$R^2 = \frac{1}{4}\left((2a-b-c)^2 + 3(c-b)^2\right) = \frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

The equilibrium is stable at a minimum of the GPE and unstable at a maximum. Since

$$\frac{\text{d}^2\text{GPE}}{\text{d}\theta^2} = \tfrac{1}{2}mg\left(-(2a-b-c)\cos\theta + (b-c)\sqrt{3}\sin\theta\right).$$

the first equilibrium position given above is unstable and the second is stable.

For the system to make complete revolutions, you need the KE at the point with maximum GPE to be greater than zero: that is, the difference in GPE between the two equilibria (which is twice the maximum GPE) is less than the KE at the point with minimum GPE.
If ω = angular velocity at stable equilibrium, you therefore require

$$\tfrac{1}{2}m\left(a^2+b^2+c^2\right)\omega^2 > mg\left((2a-b-c)\frac{(2a-b-c)}{2R} + (c-b)\sqrt{3}\frac{(c-b)\sqrt{3}}{2R}\right)$$

That is, $\left(a^2+b^2+c^2\right)\omega^2 > 2g\left(\frac{4R^2}{2R}\right) = 4gR.$