STEP III, 2005, Q10 MS

10 When the discs are a distance $2x$ apart, their centres are $2(x+r)$ apart and the length of the band is $4(x+r)+2\pi r$. Therefore the tension in the band is

$$T(x) = 2\left(\frac{\pi mg}{12}\right)\frac{4x+4r}{2\pi r} = \frac{mg}{6r}(x+r),$$

and hence the force on each disc is $F(x) = 2T(x) = \frac{mg}{3r}(x+r)$; the elastic potential energy stored in the band is

$$E(x) = \tfrac{1}{2}\left(\frac{\pi mg}{12}\right)\frac{(4x+4r)^2}{2\pi r} = \frac{mg}{3r}(x+r)^2.$$

(i) The maximum frictional resistance to the motion of a disc is μmg, so for the disc to start sliding requires $F(2r) > \mu mg$, that is $1 > \mu$. For the disc then to come to rest before a collision occurs, the elastic energy released by the band as x decreases from $2r$ to 0 must be insufficient to do the work against friction required to bring the discs into contact. This required work is $2r\mu mg$ for each disc, so $4r\mu mg$ in total, so the condition you need is $E(2r) - E(0) < 4r\mu mg$; that is $4r\mu > 3r - \frac{1}{3}r$ or $\mu > \frac{2}{3}$.

(ii) By the argument in (i), $E(2r) = E(0) + K + 4r\mu mg$, where K is the kinetic energy just before collision, so

$$K = 3rmg - \frac{1}{3}rmg - 4\mu rmg = mgr\left(\frac{8}{3} - 4\mu\right).$$

(iii) Notice first that for the discs to come to rest after the first collision, it is necessary that the discs collide, so $\mu^2 < \frac{4}{9}$, by part (i).

In order that the discs do not begin to move again, once they have come to rest for the first time after collision, each must come to rest at a point where $F(x) < \mu mg$, that is $x < (3\mu - 1)r$.

The value of x at which the particles do come to rest is given by the requirement that the loss in elastic and kinetic energy from the point of collision to the point where the discs are $2x$ apart is equal to the work done against friction on both particles in moving from 0 to $2x$ separation, that is $E(0) + \frac{1}{2}K - E(x) = 2x\mu mg$ or

$$0 = \frac{mgr}{3} + mgr\left(\frac{4}{3} - 2\mu\right) - \frac{mg}{3r}(x+r)^2 - 2x\mu$$

$$> \frac{mgr}{3} + mgr\left(\frac{4}{3} - 2\mu\right) - \frac{mg}{3r}(3\mu)^2 - 2mg\mu(3\mu-1)r$$

using the inequality on x, so

$$0 > \frac{mgr}{3}(5 - 27\mu^2)$$

or $\mu^2 > \frac{5}{27}$.