STEP III, 2004, Q9 MS

9 Let angle AOM be 2θ, so APM $= \theta$, and let R, F be the normal reaction and frictional forces of the hoop on the mouse.

The forces on the hoop are its weight, the force on the hoop from its suspension, and the reaction on the hoop to the forces R and F of the hoop on the mouse. For the hoop to be in equilibrium, the net moment of these forces about the point of suspension must be zero, but the lines of action of the weight of the hoop, and the force on it from its suspension, pass through the point of suspension, so have zero moment about it. Thus equilibrium of hoop requires the net moment of the reactions to R and F about the point of suspension of the hoop to be zero; that is, $F \times \text{PM}\cos\theta - R \times \text{PM}\sin\theta = 0$, or $F = R\tan\theta$.

For the mouse (of mass m, say) to have constant speed u, its equations of motion are:

resolving radially inward, $R - mg\cos 2\theta = \dfrac{mu^2}{a}$ and resolving tangentially, $F - mg\sin 2\theta = 0$.

Combining these three equations gives $mg\sin 2\theta\cos\theta = \left(mg\cos 2\theta + \dfrac{mu^2}{a}\right)\sin\theta$ which reduces to $u^2 = ag$ using the double angle identities.

To maintain a speed u with $u^2 = ag$ requires $R = mg(\cos 2\theta + 1) = 2mg\cos^2\theta$ and $F = mg\sin 2\theta = 2mg\cos^2\theta\tan\theta$ which is greater than μR if θ exceeds $\arctan\mu$ and hence angle AOM exceeds $2\arctan\mu$, so, initially, the hoop will begin to rotate in the opposite sense to the mouse's motion round the circle.