STEP III, 2004, Q8 MS

8 If $u = y^2$ then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{f}(x)y^2 + 2\mathrm{g}(x) = 2\mathrm{f}(x)u + 2\mathrm{g}(x)$,

which is a linear differential equation for $u(x)$.

In this case, $\mathrm{f}(x) = \frac{1}{x}$, $\mathrm{g}(x) = -1$ so the differential equation is $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2u}{x} - 2$.

The integrating factor is $\mathrm{e}^{\int \frac{2}{x}\mathrm{d}x} = \mathrm{e}^{2\ln x} = \frac{1}{x^2}$, giving $\frac{1}{x^2}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2u}{x^3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{x^2}\right) = \frac{-2}{x^2}$

so that $\frac{u}{x^2} = \int \frac{-2}{x^2}\mathrm{d}x = \frac{2}{x} + c$ or $u = y^2 = cx^2 + 2x$.

The solution curves which pass through $(1,\ 1)$, $(2,\ 2)$ and $(4,\ 4)$ are $y^2 + (x-1)^2 = 1$, $y^2 = 2x$ and $(x+2)^2 - 2y^2 = 4$ respectively. In drawing these curves it should be made clear that all of them pass through the origin, and that this is their only point of intersection; that the first is a circle with centre $(1, 0)$, the second a parabola and the third an hyperbola with centre $(-2, 0)$ and asymptotes $y = \pm\frac{x+2}{\sqrt{2}}$.